

KDV LIMIT OF THE EULER-POISSON SYSTEM

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ABSTRACT. Consider the scaling $\varepsilon^{1/2}(x - Vt) \rightarrow x$, $\varepsilon^{3/2}t \rightarrow t$ in the Euler-Poisson system for ion-acoustic waves (1.1). We establish that as $\varepsilon \rightarrow 0$, the solutions to such Euler-Poisson system converge globally in time to the solutions of the Korteweg-de Vries equation.

1. INTRODUCTION

The Euler-Poisson system is an important two-fluid model for describing the dynamics of a plasma. Consider the one dimensional Euler-Poisson system for ion-acoustic waves

$$\begin{aligned}\partial_t n + \partial_x(nu) &= 0 \\ \partial_t u + u\partial_x u + \frac{1}{M} \frac{T_i \partial_x n}{n} &= -\frac{e}{M} \partial_x \phi \\ \partial_x^2 \phi &= 4\pi e(\bar{n}e^{e\phi/T_e} - n),\end{aligned}\tag{1.1}$$

where $n(t, x)$, $u(t, x)$ and $\phi(t, x)$ are the density, velocity of the ions and the electric potential at time $t \geq 0$, position $x \in \mathbb{R}$ respectively. The parameters $e > 0$ is the electron charge, T_e is the temperature of the electron, M and T_i are the mass and temperature of the ions respectively. The electrons are described by the so called isothermal Boltzmann relation

$$n_e = \bar{n}e^{e\phi/T_e},$$

where \bar{n} is the equilibrium densities of the electrons.

Both experimental and theoretic studies show that in the long-wavelength limit, Korteweg-de Vries equation would govern the dynamics of (1.1). However, only formal derivations of such KdV limit are known [5, 9, 19, 21]. In this paper, we close this gap by justifying this limit rigourously.

1.1. Formal KdV expansion. By the classical Gardner-Morikawa transformation [19]

$$\varepsilon^{1/2}(x - Vt) \rightarrow x, \quad \varepsilon^{3/2}t \rightarrow t,\tag{1.2}$$

in (1.1), we obtain the parameterized equation

$$\begin{aligned}\varepsilon \partial_t n - V \partial_x n + \partial_x(nu) &= 0, \\ \varepsilon \partial_t u - V \partial_x u + u \partial_x u + \frac{T_i}{M} \frac{\partial_x n}{n} &= -\frac{e}{M} \partial_x \phi, \\ \varepsilon \partial_x^2 \phi &= 4\pi e(\bar{n}e^{e\phi/T_e} - n),\end{aligned}\tag{1.3}$$

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where ε is the amplitude of the initial disturbance and is assumed to be small compared with unity and V is a velocity parameter to be determined. We consider the following formal expansion

$$\begin{aligned} n &= \bar{n}(1 + \varepsilon^1 n^{(1)} + \varepsilon^2 n^{(2)} + \varepsilon^3 n^{(3)} + \varepsilon^4 n^{(4)} + \dots) \\ u &= \varepsilon^1 u^{(1)} + \varepsilon^2 u^{(2)} + \varepsilon^3 u^{(3)} + \varepsilon^4 u^{(4)} + \dots \\ \phi &= \varepsilon^1 \phi^{(1)} + \varepsilon^2 \phi^{(2)} + \varepsilon^3 \phi^{(3)} + \varepsilon^4 \phi^{(4)} + \dots \end{aligned} \quad (1.4)$$

Plugging (1.4) into (1.1), we get a power series of ε , whose coefficients depend on $(n^{(k)}, u^{(k)}, \phi^{(k)})$ for $k = 1, 2, \dots$.

The coefficients of ε^0 : The coefficient of ε^0 is automatically 0.

The coefficients of ε : Setting the coefficient of ε to be 0, we obtain

$$(\mathcal{S}_0) \begin{cases} -V \partial_x n^{(1)} + \partial_x u^{(1)} = 0 & (1.5a) \\ -V \partial_x u^{(1)} + \frac{T_i}{M} \partial_x n^{(1)} = -\frac{e}{M} \partial_x \phi^{(1)} & (1.5b) \\ 0 = \frac{e}{T_e} \phi^{(1)} - n^{(1)}. & (1.5c) \end{cases}$$

Write this equation in the matrix form

$$\begin{bmatrix} V & -1 & 0 \\ T_i/M & -V & e/M \\ 1 & 0 & -e/T_e \end{bmatrix} \begin{bmatrix} \partial_x n^{(1)} \\ \partial_x u^{(1)} \\ \partial_x \phi^{(1)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \quad (1.6)$$

In order to get a nontrivial solution for $n^{(1)}, u^{(1)}$ and $\phi^{(1)}$, we require the determinant of the coefficient matrix to vanish so that

$$\frac{T_i + T_e}{M} = V^2. \quad (1.7)$$

That is to say, we can (and need only to) adjust the velocity V , which is independent of any physical parameters, to derive the KdV equation. Furthermore, (1.5) enables us to assume the relation

$$(\mathcal{L}_1) \begin{cases} u^{(1)} = V n^{(1)}, & (1.8a) \\ \phi^{(1)} = \frac{T_e}{e} n^{(1)}, & (1.8b) \end{cases}$$

which makes (1.5) valid. Only $n^{(1)}$ still needs to be determined.

The coefficients of ε^2 and the KdV equation for $n^{(1)}$: Setting the coefficient of ε^2 to be 0, we obtain

$$(\mathcal{S}_1) \begin{cases} \partial_t n^{(1)} - V \partial_x n^{(2)} + \partial_x u^{(2)} + \partial_x (n^{(1)} u^{(1)}) = 0, & (1.9a) \\ \partial_t u^{(1)} - V \partial_x u^{(2)} + u^{(1)} \partial_x u^{(1)} \\ \quad + \frac{T_i}{M} (\partial_x n^{(2)} - n^{(1)} \partial_x n^{(1)}) = -\frac{e}{M} \partial_x \phi^{(2)}, & (1.9b) \\ \partial_x^2 \phi^{(1)} = 4\pi e \bar{n} \left(\frac{e}{T_e} \phi^{(2)} + \frac{1}{2} \left(\left(\frac{e}{T_e} \right) \phi^{(1)} \right)^2 - n^{(2)} \right). & (1.9c) \end{cases}$$

Differentiating (1.9c) with respect to x , multiplying (1.9a) with V , and (1.9c) with $T_e/(4\pi e\bar{n}M)$ respectively, and then adding them to (1.9b) together, we deduce that $n^{(1)}$ satisfies the Korteweg-de Vries equation

$$\partial_t n^{(1)} + V n^{(1)} \partial_x n^{(1)} + \frac{1}{2} \frac{T_e}{4\pi \bar{n} e M V} \partial_x^3 n^{(1)} = 0, \quad (1.10)$$

where we have used the relation (1.8) and (1.7), under which all the coefficients of $n^{(2)}, u^{(2)}$ and $\phi^{(2)}$ vanish. We also note that the system (1.10) and (1.8) for $(n^{(1)}, u^{(1)}, \phi^{(1)})$ are self contained, which do not depend on $(n^{(j)}, u^{(j)}, \phi^{(j)})$ for $j \geq 2$. The above formal derivation for the case $T_i = 0$ can be found in [19]; while the derivation for the case $T_i > 0$ is new.

Now we want to find out the equations satisfied by $(n^{(2)}, u^{(2)}, \phi^{(2)})$ assuming that $(n^{(1)}, u^{(1)}, \phi^{(1)})$ is known (solved from (1.10) and (1.8)). From (1.9), we can express $(n^{(2)}, u^{(2)}, \phi^{(2)})$ in terms of $(n^{(1)}, u^{(1)}, \phi^{(1)})$:

$$(\mathcal{L}_2) \begin{cases} \phi^{(2)} = \frac{T_e}{e} (n^{(2)} + h^{(1)}), & h^{(1)} = \frac{1}{4\pi e \bar{n}} \partial_x^2 \phi^{(1)} - \frac{1}{2} \left(\frac{e}{T_e} \phi^{(1)} \right)^2, & (1.11a) \\ u^{(2)} = V n^{(2)} + g^{(1)}, & g^{(1)} = \int^x \mathfrak{g}^{(1)}(t, \xi) d\xi, & (1.11b) \\ & \mathfrak{g}^{(1)} = -\partial_t n^{(1)} + \partial_x (n^{(1)} u^{(1)}), \end{cases}$$

which make (1.9) valid. Only $n^{(2)}$ needs to be determined now.

The coefficients of ε^3 and the linearized KdV equation for $n^{(2)}$: Setting the coefficient of ε^3 to be zero, we obtain

$$(\mathcal{S}_2) \begin{cases} \partial_t n^{(2)} - V \partial_x n^{(3)} + \partial_x u^{(3)} + \partial_x (n^{(1)} u^{(2)} + n^{(2)} u^{(1)}) = 0, & (1.12a) \\ \partial_t u^{(2)} - V \partial_x u^{(3)} + \partial_x (u^{(1)} u^{(2)}) + \frac{T_i}{M} \partial_x n^{(3)} \\ \quad - \frac{T_i}{M} [\partial_x (n^{(1)} n^{(2)}) - (n^{(1)})^2 \partial_x n^{(1)}] = -\frac{e}{M} \partial_x \phi^{(3)}, & (1.12b) \\ \partial_x^2 \phi^{(2)} = 4\pi e \bar{n} \left[\frac{e}{T_e} \phi^{(3)} + \left(\frac{e}{T_e} \right)^2 \phi^{(1)} \phi^{(2)} + \frac{1}{3!} \left(\frac{e}{T_e} \phi^{(1)} \right)^3 - n^{(3)} \right]. & (1.12c) \end{cases}$$

Differentiating (1.12c) with respect to x , multiplying (1.12a) with V , and (1.12c) with $T_e/(4\pi e \bar{n} M)$ respectively, and then adding them to (1.12b) together, we deduce that $n^{(2)}$ satisfies the linearized inhomogeneous Korteweg-de Vries equation

$$\partial_t n^{(2)} + V \partial_x (n^{(1)} n^{(2)}) + \frac{1}{2} \frac{T_e}{4\pi \bar{n} e M V} \partial_x^3 n^{(2)} = G^{(1)}, \quad (1.13)$$

where we have used (1.11) and $G^{(1)} = G^{(1)}(n^{(1)})$ depends only on $n^{(1)}$. Again, the system (1.13) and (1.11) for $(n^{(2)}, u^{(2)}, \phi^{(2)})$ are self contained and do not depend on $(n^{(j)}, u^{(j)}, \phi^{(j)})$ for $j \geq 3$.

The coefficients of ε^{k+1} and the linearized KdV equation for $n^{(k)}$: Let $k \geq 3$ be an integer. Recalling that in the k^{th} step, by setting the coefficient of ε^k to be 0, we

obtain an evolution system (\mathcal{S}_{k-1}) for $(n^{(k-1)}, u^{(k-1)}, \phi^{(k-1)})$, from which we obtain

$$(\mathcal{L}_k) \begin{cases} \phi^{(k)} = \frac{T_e}{e}(n^{(k)} + h^{(k-1)}), \text{ for some } h^{(k-1)} \text{ depending} & (1.14a) \\ \text{only on } (n^j, u^j, \phi^j) \text{ for } 1 \leq j \leq k-1, \\ u^{(k)} = Vn^{(k)} + g^{(k-1)}, \quad g^{(k-1)} = \int^x \mathbf{g}^{(k-1)}(t, \xi) d\xi, \text{ for some} & (1.14b) \\ \mathbf{g}^{(k-1)} \text{ depending only on } (n^j, u^j, \phi^j) \text{ for } 1 \leq j \leq k-1. \end{cases}$$

This relation makes (\mathcal{S}_{k-1}) valid, and we need only to determine $n^{(k)}$. By setting the coefficient of ε^{k+1} to be 0, we obtain an evolution system (\mathcal{S}_k) for $(n^{(k)}, u^{(k)}, \phi^{(k)})$. By the same procedure that leads to (1.13), we obtain the linearized inhomogeneous Korteweg-de Vries equation for $n^{(k)}$

$$\partial_t n^{(k)} + V \partial_x (n^{(1)} n^{(k)}) + \frac{1}{2} \frac{T_e}{4\pi\bar{n}eMV} \partial_x^3 n^{(k)} = G^{(k-1)}, \quad k \geq 3, \quad (1.15)$$

where $G^{(k-1)}$ depends only on $n^{(1)}, \dots, n^{(k-1)}$, which are “known” from the first $(k-1)^{th}$ steps. Again, the system (1.14) and (1.15) for $(n^{(k)}, u^{(k)}, \phi^{(k)})$ are self contained, which do not depend on $(n^{(j)}, u^{(j)}, \phi^{(j)})$ for $j \geq k+1$.

For the solvability of $(n^{(k)}, u^{(k)}, \phi^{(k)})$ for $k \geq 1$, we have the following two theorems.

Theorem 1.1. *Let $\tilde{s}_1 \geq 2$ be a sufficiently large integer. Then for any given initial data $n_0^{(1)} \in H^{\tilde{s}_1}(\mathbb{R})$, there exists $\tau_* > 0$ such that the initial value problem (1.10) and (1.8) has a unique solution*

$$(n^{(1)}, u^{(1)}, \phi^{(1)}) \in L^\infty(-\tau_*, \tau_*; H^{\tilde{s}_1}(\mathbb{R}))$$

with initial data $(n_0^{(1)}, Vn_0^{(1)}, T_e n_0^{(1)}/e)$. Furthermore, by using the conservation laws of the KdV equation, we can extend the solution to any time interval $[-\tau, \tau]$.

This result is classical for the KdV equation, see for example [10]. See also [11, 16] for more details on KdV equation.

Theorem 1.2. *Let $k \geq 2$ and $\tilde{s}_k \leq \tilde{s}_1 - 3(k-1)$ be a sufficiently large integer. Then for any $\tau > 0$ and any given initial data $(n_0^{(k)}, u_0^{(k)}, \phi_0^{(k)}) \in H^{\tilde{s}_k}(\mathbb{R})$, the initial value problem (1.15) and (1.14) with initial data $(n_0^{(k)}, u_0^{(k)}, \phi_0^{(k)})$ satisfying (1.14) has a unique solution*

$$(n^{(k)}, u^{(k)}, \phi^{(k)}) \in L^\infty(-\tau, \tau; H^{\tilde{s}_k}(\mathbb{R})).$$

The proof of Theorem 1.2 is standard. See Appendix. In the following, we will assume that these solutions $(n^{(k)}, u^{(k)}, \phi^{(k)})$ for $1 \leq k \leq 4$ are sufficiently smooth. The optimality of \tilde{s}_k will not be addressed in this paper.

1.2. Main result. To show that $n^{(1)}$ converges to a solution of the KdV equation as $\varepsilon \rightarrow 0$, we must make the above procedure rigorous. Let (n, u, ϕ) be a solution of

the scaled system (1.3) of the following expansion

$$\begin{aligned} n &= \bar{n}(1 + \varepsilon^1 n^{(1)} + \varepsilon^2 n^{(2)} + \varepsilon^3 n^{(3)} + \varepsilon^4 n^{(4)} + \varepsilon^3 n_R^\varepsilon) \\ u &= \varepsilon^1 u^{(1)} + \varepsilon^2 u^{(2)} + \varepsilon^3 u^{(3)} + \varepsilon^4 u^{(4)} + \varepsilon^3 u_R^\varepsilon \\ \phi &= \varepsilon^1 \phi^{(1)} + \varepsilon^2 \phi^{(2)} + \varepsilon^3 \phi^{(3)} + \varepsilon^4 \phi^{(4)} + \varepsilon^3 \phi_R^\varepsilon, \end{aligned} \quad (1.16)$$

where $(n^{(1)}, u^{(1)}, \phi^{(1)})$ satisfies (1.8) and (1.10), $(n^{(k)}, u^{(k)}, \phi^{(k)})$ satisfies (1.14) and (1.15) for $2 \leq k \leq 4$, and $(n_R^\varepsilon, u_R^\varepsilon, \phi_R^\varepsilon)$ is the remainder.

In the following, we derive the remainder system satisfied by $(n_R^\varepsilon, u_R^\varepsilon, \phi_R^\varepsilon)$. To simplify the expression, we denote

$$\tilde{n} = n^{(1)} + \varepsilon n^{(2)} + \varepsilon^2 n^{(3)} + \varepsilon^3 n^{(4)}, \quad \tilde{u} = u^{(1)} + \varepsilon u^{(2)} + \varepsilon^2 u^{(3)} + \varepsilon^3 u^{(4)}.$$

After careful computations (see Appendix for details), we obtain the following remainder system for $(n_R^\varepsilon, u_R^\varepsilon, \phi_R^\varepsilon)$:

$$\begin{cases} \partial_t n_R^\varepsilon - \frac{V-u}{\varepsilon} \partial_x n_R^\varepsilon + \frac{n}{\varepsilon} \partial_x u_R^\varepsilon + \partial_x \tilde{n} u_R^\varepsilon + \partial_x \tilde{u} n_R^\varepsilon + \varepsilon \mathcal{R}_1 = 0, \end{cases} \quad (1.17a)$$

$$\begin{cases} \partial_t u_R^\varepsilon - \frac{V-u}{\varepsilon} \partial_x u_R^\varepsilon + \frac{1}{\varepsilon} \frac{T_i}{M} \partial_x n_R^\varepsilon - \frac{T_i}{M} \left(\frac{\tilde{n} + \varepsilon n_R^\varepsilon}{n} \right) \partial_x n_R^\varepsilon \\ \quad + \partial_x \tilde{u} u_R^\varepsilon - \frac{T_i}{M} \frac{b}{n} n_R^\varepsilon + \varepsilon \mathcal{R}_2 = -\frac{1}{\varepsilon} \frac{e}{M} \partial_x \phi_R^\varepsilon, \end{cases} \quad (1.17b)$$

$$\begin{cases} \varepsilon \partial_x^2 \phi_R^\varepsilon = 4\pi e \bar{n} \left[\frac{e}{T_e} \phi_R^\varepsilon + \varepsilon \left(\frac{e}{T_e} \right)^2 \phi^{(1)} \phi_R^\varepsilon - n_R^\varepsilon \right] + \varepsilon^2 \mathcal{R}_3, \end{cases} \quad (1.17c)$$

where

$$\begin{cases} b = \partial_x n^{(1)} + \varepsilon (\partial_x n^{(2)} - n^{(1)} \partial_x n^{(1)}) \\ \quad + \varepsilon^2 (\partial_x n^{(3)} + [(n^{(1)})^2 - \partial_x n^{(1)}] \partial_x n^{(1)} + n^{(1)} \partial_x n^{(2)}) \\ \quad + \varepsilon^3 (\partial_x n^{(4)} - [\partial_x (n^{(1)} n^{(3)}) + (n^{(2)} - (n^{(1)})^2) \partial_x n^{(2)} \\ \quad + ((n^{(1)})^3 - 2n^{(1)} n^{(2)}) \partial_x n^{(1)}]), \end{cases} \quad (1.18a)$$

$$\begin{cases} \mathcal{R}_1 = \partial_t n^{(4)} + \sum_{1 \leq i, j \leq 4; i+j \geq 5} \varepsilon^{i+j-5} \partial_x (n^{(i)} u^{(j)}), \end{cases} \quad (1.18b)$$

$$\begin{cases} \mathcal{R}_2 = \partial_t u^{(4)} + \sum_{1 \leq i, j \leq 4; i+j \geq 5} \varepsilon^{i+j-5} u^{(i)} \partial_x u^{(j)} + \frac{T_i}{M} \frac{1}{n} \{ \text{finite} \\ \quad \text{combination of } n^{(i)} (1 \leq i \leq 4) \text{ and their derivatives} \}, \end{cases} \quad (1.18c)$$

$$\begin{cases} \mathcal{R}_3 = \left[\frac{1}{2} \left(\frac{e}{T_e} \right)^2 (\varepsilon \phi_R^\varepsilon) + \left(\frac{e}{T_e} \right)^2 (\phi^{(2)} + \frac{1}{2} \frac{e}{T_e} (\phi^{(1)})^2) \right] \phi_R^\varepsilon + \hat{R}'(\varepsilon \phi_R^\varepsilon). \end{cases} \quad (1.18d)$$

One can refer to the Appendix for the detailed derivation of \mathcal{R}_3 , which is a smooth function of ϕ_R^ε . In particular, \mathcal{R}_3 does not involve any derivatives of ϕ_R^ε . The mathematical key difficulty is to derive estimates for the remainders $(n_R^\varepsilon, u_R^\varepsilon, \phi_R^\varepsilon)$ uniformly in ε .

Our main result of this paper is the following

Theorem 1.3. *Let $\tilde{s}_i \geq 2$ in Theorem 1.1 and 1.2 be sufficiently large and $(n^{(1)}, u^{(1)}, \phi^{(1)}) \in H^{\tilde{s}_1}$ be a solution constructed in Theorem 1.1 for the KdV equation*

with initial data $(n_0^{(1)}, u_0^{(1)}, \phi_0^{(1)}) \in H^{\tilde{s}_1}$ satisfying (1.8). Let $(n^{(i)}, u^{(i)}, \phi^{(i)}) \in H^{\tilde{s}_i}$ ($i = 2, 3, 4$) be solutions of (1.15) and (1.14) constructed in Theorem 1.2 with initial data $(n_0^{(i)}, u_0^{(i)}, \phi_0^{(i)}) \in H^{\tilde{s}_i}$ satisfying (1.14). Let $(n_{R0}^\varepsilon, u_{R0}^\varepsilon, \phi_{R0}^\varepsilon)$ satisfy (1.17) and assume

$$\begin{aligned} n_0 &= \bar{n}(1 + \varepsilon^1 n_0^{(1)} + \varepsilon^2 n_0^{(2)} + \varepsilon^3 n_0^{(3)} + \varepsilon^4 n_0^{(4)} + \varepsilon^3 n_{R0}^\varepsilon), \\ u_0 &= \varepsilon^1 u_0^{(1)} + \varepsilon^2 u_0^{(2)} + \varepsilon^3 u_0^{(3)} + \varepsilon^4 u_0^{(4)} + \varepsilon^3 u_{R0}^\varepsilon, \\ \phi_0 &= \varepsilon^1 \phi_0^{(1)} + \varepsilon^2 \phi_0^{(2)} + \varepsilon^3 \phi_0^{(3)} + \varepsilon^4 \phi_0^{(4)} + \varepsilon^3 \phi_{R0}^\varepsilon. \end{aligned}$$

Then for any $\tau > 0$, there exists $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$, the solution of the EP system (1.3) with initial data (n_0, u_0, ϕ_0) can be expressed as

$$\begin{aligned} n &= \bar{n}(1 + \varepsilon^1 n^{(1)} + \varepsilon^2 n^{(2)} + \varepsilon^3 n^{(3)} + \varepsilon^4 n^{(4)} + \varepsilon^3 n_R^\varepsilon), \\ u &= \varepsilon^1 u^{(1)} + \varepsilon^2 u^{(2)} + \varepsilon^3 u^{(3)} + \varepsilon^4 u^{(4)} + \varepsilon^3 u_R^\varepsilon, \\ \phi &= \varepsilon^1 \phi^{(1)} + \varepsilon^2 \phi^{(2)} + \varepsilon^3 \phi^{(3)} + \varepsilon^4 \phi^{(4)} + \varepsilon^3 \phi_R^\varepsilon, \end{aligned}$$

such that for all $0 < \varepsilon < \varepsilon_0$,

1) when $T_i > 0$,

$$\sup_{[0, \tau]} \|(n_R^\varepsilon, u_R^\varepsilon, \phi_R^\varepsilon)\|_{H^2}^2 \leq C_\tau (1 + \|(n_{R0}^\varepsilon, u_{R0}^\varepsilon, \phi_{R0}^\varepsilon)\|_{H^2}^2),$$

2) when $T_i = 0$,

$$\begin{aligned} &\sup_{[0, \tau]} \{ \|(n_R^\varepsilon, u_R^\varepsilon, \phi_R^\varepsilon)\|_{H^2}^2 + \varepsilon \|(\partial_x^3 u_R^\varepsilon, \partial_x^3 \phi_R^\varepsilon)\|_{L^2}^2 + \varepsilon^2 \|\partial_x^4 \phi_R^\varepsilon\|_{L^2}^2 \} \\ &\leq C_\tau (1 + \|(n_{R0}^\varepsilon, u_{R0}^\varepsilon, \phi_{R0}^\varepsilon)\|_{H^2}^2 + \varepsilon \|(\partial_x^3 u_{R0}^\varepsilon, \partial_x^3 \phi_{R0}^\varepsilon)\|_{L^2}^2 + \varepsilon^2 \|\partial_x^4 \phi_{R0}^\varepsilon\|_{L^2}^2). \end{aligned}$$

Remark 1.4. While we get a global uniform in ε estimate for the H^2 norm of the remainders, the H^3 norm or the H^4 norm may blow up in finite time. However, they are both uniformly bounded after multiplied by $\varepsilon^{1/2}$ and ε respectively.

Our result provides a rigorous and unified justification of the KdV equation limit of the Euler-Poisson system for ion-acoustic waves with Boltzmann relation. The classical formal derivation in [19] deals with only the case of $T_i = 0$, while our results cover all the case of $T_i \geq 0$. When $T_i > 0$, the control of the remainder falls into the framework of Grenier [3], where the author studied some singular limits by using the pseudo-differential operator (PsDO) techniques for singular perturbations of hyperbolic systems. But suitable decomposition of (1.17c) is required.

Unfortunately, in the classical case of $T_i = 0$, we cannot apply Grenier's machinery to get uniform estimate for the remainders. This is because when $T_i = 0$, the matrix P_ε^{-1} given by (2.21) is not a bounded family of PsDOs of order 0 any more, see [3] or [18] for more details on PsDO theory. To overcome this difficulty, we need to employ a careful combination of delicate energy estimate together with analysis of the structure of the remainder system.

The basic plan is to first estimate some uniform bound for $(u_R^\varepsilon, \phi_R^\varepsilon)$ and then recover the estimate for n_R^ε from the estimate of ϕ_R^ε by the Poisson equation (1.17c)

(see Lemma 3.1). We want to apply the Gronwall lemma to complete the proof. To state clearly, we first define (see (3.2))

$$\|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2 = \|u_R^\varepsilon\|_{H^2}^2 + \|\phi_R^\varepsilon\|_{H^2}^2 + \varepsilon \|\partial_x^3 u_R^\varepsilon\|^2 + \varepsilon \|\partial_x^3 \phi_R^\varepsilon\|^2 + \varepsilon^2 \|\partial_x^4 \phi_R^\varepsilon\|^2. \quad (1.19)$$

As we will see, the zeroth order, the first to the second order estimates for $(u_R^\varepsilon, \phi_R^\varepsilon)$ can be controlled in terms of $\|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2$. Unfortunately, the third order estimate of $\sqrt{\varepsilon}(u_R^\varepsilon, \phi_R^\varepsilon)$ involves a bad term $\mathcal{B}^{(3 \times \varepsilon)}$ (see (3.49) and Remark 3.8)

$$\mathcal{B}^{(3 \times \varepsilon)} = - \int \partial_x^3 \phi_R^\varepsilon \partial_x \left[\frac{\varepsilon^2}{n} \right] \partial_t \partial_x^4 \phi_R^\varepsilon dx, \quad (1.20)$$

where $\partial_t \partial_x^4 \phi_R^\varepsilon$ cannot be controlled in terms of $\|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon$. Even worse, this difficulty persists no matter how high the Sobolev order or the expansion order is. For example, when we want to estimate the H^k norm of $(u_R^\varepsilon, \phi_R^\varepsilon)$, we get a term $\mathcal{B}^{(k)} = - \int \partial_x^k \phi_R^\varepsilon \partial_x \left[\frac{\varepsilon}{n} \right] \partial_t \partial_x^{k+1} \phi_R^\varepsilon dx$ with the same structure of (1.20). Note that $\mathcal{B}^{(3 \times \varepsilon)}$ in equation (3.49) is just $\mathcal{B}^{(3)}$ multiplied by ε .

Fortunately, we are able to employ the precise structure of (1.17c) to overcome such a difficulty. In the second order estimate, we can extract a precise term $\mathcal{B}^{(2)}$ (after integration by parts, see (3.26))

$$\mathcal{B}^{(2)} = \int \partial_x^3 \phi_R^\varepsilon \partial_x \left[\frac{\varepsilon}{n} \right] \partial_t \partial_x^2 \phi_R^\varepsilon.$$

Even though $\partial_t \partial_x^4 \phi_R^\varepsilon$ in (1.20) is out of control, the combination of

$$\partial_t \partial_x^2 \phi_R^\varepsilon - \varepsilon \partial_t \partial_x^4 \phi_R^\varepsilon$$

can be controlled in terms of $\|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2$ by using the Poisson equation (1.17c).

In recent years, there have been a large number of studies of the Euler-Poisson (Maxwell) equation and related various singular limit [1, 2, 4, 6–8, 12–14, 20]. In [17], KdV equation is derived rigorously from the water-wave equation.

This paper is organized as follows. In Section 2, we prove the limit for the case of $T_i > 0$, by using the PsDO framework of Grenier [3]. In Section 3, we prove the limit for the classical case of $T_i = 0$, where more delicate estimate is required. Throughout this paper, $\|\cdot\|$ denotes the L^2 norm.

2. UNIFORM ENERGY ESTIMATES: THE CASE $T_i > 0$

In this section, we give the energy estimates uniformly in ε for the case of $T_i > 0$ for the remainder system (1.17) of $(n_R^\varepsilon, u_R^\varepsilon, \phi_R^\varepsilon)$. This section is divided into two parts. In the first one, we introduce an abstract form of the remainder equations of n_R^ε and u_R^ε , while ϕ_R^ε is only included implicitly. This abstract form is more suitable for us to apply the PsDO framework in Grenier [3]. Then in the second part, we establish energy estimates and prove the main theorem for $T_i > 0$. We remark that PsDO framework is applicable mainly because this system is symmetrizable when $T_i > 0$, see also [15].

For notational convenience, we normalize the physical constants e, M, T_i, T_e to be 1 and $\bar{n} = (4\pi e)^{-1}$ in (1.17) throughout this section. Therefore, $V = \sqrt{2}$ by (1.6). Let $\tau \geq 0$ be arbitrarily fixed, we will establish estimates in $L^\infty(0, \tau; H^{s'})$ for any $2 \leq s' \leq \tilde{s}_4 - 3$, where \tilde{s}_4 is sufficiently large and fixed in Theorem 1.1.

2.1. Reduction. We follow Grenier's framework of [3] (see also [1]). Before we give the uniform estimate, we first reduce (1.17) into an abstract form.

Lemma 2.1. *Let $(n_R^\varepsilon, u_R^\varepsilon, \phi_R^\varepsilon)$ be a solution to (1.17) and $w = (n_R^\varepsilon, u_R^\varepsilon)^T$. Then w satisfies the following system*

$$w_t + \mathcal{A}_\varepsilon(w)w + \mathcal{R}(w) = 0, \quad (2.1)$$

where \mathcal{A}_ε is a family of pseudodifferential operators whose symbol depends on the solution w , and can be decomposed into the sum of a “regular” part and a “singular” part

$$\mathcal{A}_\varepsilon(w) = \mathcal{A}_{1,\varepsilon}(w) + \mathcal{A}_{2,\varepsilon},$$

whose symbols are respectively the following matrices

$$A_{1,\varepsilon}(w) = \begin{bmatrix} i\xi(\tilde{u} + \varepsilon^2 u_R^\varepsilon) & i\xi(\tilde{n} + \varepsilon^2 n_R^\varepsilon) \\ -\frac{i\xi(\tilde{n} + \varepsilon^2 n_R^\varepsilon)}{n} - \frac{i\xi\phi^{(1)}}{(1+\varepsilon\xi^2)(1+\varepsilon\phi^{(1)}+\varepsilon\xi^2)} & i\xi(\tilde{u} + \varepsilon^2 u_R^\varepsilon) \end{bmatrix} \quad (2.2)$$

and

$$A_{2,\varepsilon} = \frac{1}{\varepsilon} \tilde{A}_{2,\varepsilon}(\xi) = \frac{1}{\varepsilon} \begin{bmatrix} -\sqrt{2}i\xi & i\xi \\ i\xi + \frac{i\xi}{1+\varepsilon\xi^2} & -\sqrt{2}i\xi \end{bmatrix}. \quad (2.3)$$

In (2.1),

$$\mathcal{R}(w) = \mathcal{F}_\varepsilon(w)w + \mathcal{N}(w), \quad (2.4)$$

where $\mathcal{F}_\varepsilon(w)$ is the coefficient matrix before $(n_R^\varepsilon, u_R^\varepsilon)^T$:

$$\mathcal{F}_\varepsilon(w) = \begin{bmatrix} \partial_x \tilde{u} & \partial_x \tilde{n} \\ -\frac{b}{n} & \partial_x \tilde{u} \end{bmatrix}, \quad (2.5)$$

and $\mathcal{N} = [\mathcal{N}_1, -\mathcal{N}_2]^T$ is defined by (2.19). There exists constant C_α

$$\|\mathcal{F}_\varepsilon(w)w\|_{H^\alpha} + \|\mathcal{N}(w)\|_{H^\alpha} \leq C_\alpha(1 + \|n_R^\varepsilon\| + \|u_R^\varepsilon\|) \quad (2.6)$$

for every $2 \leq \alpha \leq s'$.

Proof. To reduce the system (1.17) to an evolution system for $(n_R^\varepsilon, u_R^\varepsilon)$, we need to express ϕ_R^ε in terms of n_R^ε and u_R^ε . We therefore consider the decomposition

$$\phi_R^\varepsilon = \Phi_1 + \Phi_2 + \Phi_3, \quad (2.7)$$

where Φ_1, Φ_2 and Φ_3 are specified below.

Recall (1.17c),

$$\varepsilon \partial_x^2 \phi_R^\varepsilon = (1 + \varepsilon \phi^{(1)}) \phi_R^\varepsilon - n_R^\varepsilon + \varepsilon^2 \mathcal{R}_3. \quad (2.8)$$

First, we define

$$\Phi_1 = Op\left(\frac{1}{(1 + \varepsilon \phi^{(1)}) + \varepsilon \xi^2}\right) n_R^\varepsilon,$$

where $Op\left(\frac{1}{(1 + \varepsilon \phi^{(1)}) + \varepsilon \xi^2}\right)$ is a PsDO with limited smoothness (see [3] for more details), for all $0 < \varepsilon < \varepsilon_1$ for some $\varepsilon_1 > 0$. We then have

$$\|\Phi_1\|_{H^\alpha} \leq C \|n_R^\varepsilon\|_{H^\alpha}. \quad (2.9)$$

In fact, by standard PsDO calculus, we have

$$Op((1 + \varepsilon \phi^{(1)}) + \varepsilon \xi^2) Op\left(\frac{1}{(1 + \varepsilon \phi^{(1)}) + \varepsilon \xi^2}\right) n_R^\varepsilon = n_R^\varepsilon + \varepsilon \tilde{\mathcal{S}}_1 n_R^\varepsilon,$$

where $\tilde{\mathcal{S}}_1$ is a bounded operator from H^α to $H^{\alpha+1}$ defined by

$$\begin{aligned}\tilde{\mathcal{S}}_1 &= \frac{1}{\varepsilon} \left((1 + \varepsilon\phi^{(1)})Op\left(\frac{1}{1 + \varepsilon\phi^{(1)} + \varepsilon\xi^2}\right) - Op\left(\frac{1 + \varepsilon\phi^{(1)}}{1 + \varepsilon\phi^{(1)} + \varepsilon\xi^2}\right) \right) \\ &= \phi^{(1)}Op\left(\frac{1}{1 + \varepsilon\phi^{(1)} + \varepsilon\xi^2}\right) - Op\left(\frac{\phi^{(1)}}{1 + \varepsilon\phi^{(1)} + \varepsilon\xi^2}\right).\end{aligned}$$

We remark that the ε in front of $\tilde{\mathcal{S}}_1$ is very important, since it cancels part of the singularity of $\varepsilon^{-1}\partial_x\phi_R^\varepsilon$ in (1.17). Equivalently, Φ_1 is a solution of

$$\varepsilon\partial_x^2\Phi_1 = (1 + \varepsilon\phi^{(1)})\Phi_1 - n_R^\varepsilon - \varepsilon\tilde{\mathcal{S}}_1 n_R^\varepsilon. \quad (2.10)$$

This enables us to define Φ_2 to be the solution of

$$\varepsilon\partial_x^2\Phi_2 = (1 + \varepsilon\phi^{(1)})\Phi_2 + \varepsilon\tilde{\mathcal{S}}_1 n_R^\varepsilon. \quad (2.11)$$

It is straightforward that for $\alpha \geq 1$

$$\|\Phi_2\|_{H^\alpha} \leq \varepsilon C \|\tilde{\mathcal{S}}_1 n_R^\varepsilon\|_{H^\alpha} \leq \varepsilon C \|n_R^\varepsilon\|_{H^{\alpha-1}}. \quad (2.12)$$

Finally, we define Φ_3 to be the solution of

$$\varepsilon\partial_x^2\Phi_3 = (1 + \varepsilon\phi^{(1)})\Phi_3 + \varepsilon^2\mathcal{R}_3. \quad (2.13)$$

By superposition of linear equations (2.10), (2.11) and (2.13), we get (2.7).

Now, we consider the decomposition of $-\frac{1}{\varepsilon}\partial_x\phi_R^\varepsilon$ on the RHS of (1.17b). For the expression of \mathcal{R}_3 in (1.18d), by Lemma A.1 there exists constant $C = C(\|\phi^{(i)}\|_{H^{\tilde{s}_i}}, \varepsilon\|\phi_R^\varepsilon\|_{H^\alpha})$ such that

$$\begin{aligned}\|\mathcal{R}_3\|_{H^\alpha} &\leq C\|\phi_R^\varepsilon\|_{H^\alpha} \\ &\leq C(\|\Phi_1\|_{H^\alpha}, \|\Phi_2\|_{H^\alpha}, \|\Phi_3\|_{H^\alpha}),\end{aligned} \quad (2.14)$$

for any α such that $2 \leq \alpha \leq s'$, for some $s' \leq \tilde{s}_4$ depending on \tilde{s}_4 . Taking inner product of (2.13) with $\partial_x^\alpha\Phi_3$ and integrating by parts, we have

$$\varepsilon\|\partial_x^{\alpha+1}\Phi_3\|^2 + \int \partial_x^\alpha((1 + \varepsilon\phi^{(1)})\Phi_3)\partial_x^\alpha\Phi_3 \leq \varepsilon^2 C(\|\Phi_1\|_{H^\alpha}, \|\Phi_2\|_{H^\alpha}, \|\Phi_3\|_{H^\alpha})\|\partial_x^\alpha\Phi_3\|.$$

On the other hand, since $\|\varepsilon\phi^{(1)}\|_{L^\infty} < 1/2$ when $0 < \varepsilon < \varepsilon_1$ for some $\varepsilon_1 > 0$, we obtain

$$\varepsilon\|\partial_x\Phi_3\|_{H^\alpha} + \|\Phi_3\|_{H^\alpha} \leq \varepsilon^2 C(\|\Phi_1\|_{H^\alpha}, \|\Phi_2\|_{H^\alpha}). \quad (2.15)$$

Therefore, from (2.12), (2.15) and (2.9),

$$\begin{aligned}\left\|\frac{1}{\varepsilon}(\partial_x\Phi_2 + \partial_x\Phi_3)\right\|_{H^\alpha} &\leq \left\|\frac{1}{\varepsilon}\partial_x\Phi_2\right\|_{H^\alpha} + \left\|\frac{1}{\varepsilon}\partial_x\Phi_3\right\|_{H^\alpha} \\ &\leq C\|n_R^\varepsilon\|_{H^\alpha} + C(\|\Phi_1\|_{H^\alpha}, \|\Phi_2\|_{H^\alpha}) \\ &\leq C(\|n_R^\varepsilon\|_{H^\alpha}).\end{aligned} \quad (2.16)$$

On the other hand, by symbolic calculus, we have

$$\begin{aligned}-\frac{1}{\varepsilon}\partial_x\Phi_1 &= -\frac{1}{\varepsilon}Op(i\xi)Op\left(\frac{1}{(1 + \varepsilon\phi^{(1)}) + \varepsilon\xi^2}\right)n_R^\varepsilon \\ &= -\frac{1}{\varepsilon}Op\left(\frac{i\xi}{(1 + \varepsilon\phi^{(1)}) + \varepsilon\xi^2}\right)n_R^\varepsilon + \mathcal{S}_2 n_R^\varepsilon,\end{aligned}$$

where

$$\mathcal{S}_2 = Op\left(\frac{\partial_x \phi^{(1)}}{(1 + \varepsilon \phi^{(1)} + \varepsilon \xi^2)^2}\right) \quad (2.17)$$

is a bounded operator from H^α to H^α for every $\alpha \leq s'$. Recalling (2.7), we obtain the decomposition of $-\frac{1}{\varepsilon} \partial_x \phi_R^\varepsilon$ on the RHS of (1.17b):

$$-\frac{1}{\varepsilon} \partial_x \phi_R^\varepsilon = -\frac{1}{\varepsilon} Op\left(\frac{i\xi}{(1 + \varepsilon \phi^{(1)} + \varepsilon \xi^2)}\right) n_R^\varepsilon + \mathcal{S}_2 n_R^\varepsilon - \frac{1}{\varepsilon} (\partial_x \Phi_2 + \partial_x \Phi_3). \quad (2.18)$$

Defining

$$\mathcal{N}_1 = -\varepsilon \mathcal{R}_1, \quad \mathcal{N}_2 = \mathcal{S}_2 n_R^\varepsilon - \frac{1}{\varepsilon} (\partial_x \Phi_2 + \partial_x \Phi_3) - \varepsilon \mathcal{R}_2, \quad (2.19)$$

where \mathcal{R}_1 and \mathcal{R}_2 are defined in (1.17b) and (1.17c) respectively, we can transform the remainder system (1.17) into the abstract form (2.1). Note also that from (2.16) and (2.17), \mathcal{N} is bounded by (2.6). \square

2.2. Energy estimates. In this subsection, we will complete the proof of Theorem 1.3 for the case $T_i > 0$. For this, we need only uniform energy estimates for (2.1), where the matrices $A_{1,\varepsilon}$ and $A_{1,\varepsilon}$ are given by (2.2) and (2.3) respectively. To further simplify the notations, we denote

$$\begin{aligned} N_R &= \tilde{n} + \varepsilon^2 n_R^\varepsilon; & U_R &= \tilde{u} + \varepsilon^2 u_R^\varepsilon, \\ n_1 &= n = 1 + \varepsilon N_R, & n_2 &= 1 + \varepsilon \phi^{(1)} + \varepsilon \xi^2. \end{aligned} \quad (2.20)$$

In these notations,

$$A_\varepsilon = i\xi \begin{bmatrix} (U_R - \frac{1}{\varepsilon}) & \frac{n_1}{\varepsilon} \\ \frac{1}{\varepsilon(1+\varepsilon N_R)} + \frac{1}{\varepsilon n_2} & (U_R - \frac{1}{\varepsilon}) \end{bmatrix},$$

whose eigenvalues are

$$\lambda_\pm = i\xi \left((U_R - \frac{1}{\varepsilon}) \pm \frac{\sqrt{n_1}}{\varepsilon} \frac{\sqrt{n_2 + n_1}}{\sqrt{n_1 n_2}} \right)$$

and their normalized eigenvectors are

$$e_\pm = \begin{bmatrix} \frac{n_1 \sqrt{n_2}}{\sqrt{n_1^2 n_2 + n_2 + n_1}} \\ \pm \frac{\sqrt{n_2 + n_1}}{\sqrt{n_1^2 n_2 + n_2 + n_1}} \end{bmatrix}.$$

Let

$$\begin{aligned} P_\varepsilon &= \begin{bmatrix} \frac{n_1 \sqrt{n_2}}{\sqrt{n_1^2 n_2 + n_2 + n_1}} & \frac{n_1 \sqrt{n_2}}{\sqrt{n_1^2 n_2 + n_2 + n_1}} \\ \frac{\sqrt{n_2 + n_1}}{\sqrt{n_1^2 n_2 + n_2 + n_1}} & -\frac{\sqrt{n_2 + n_1}}{\sqrt{n_1^2 n_2 + n_2 + n_1}} \end{bmatrix}, \\ P_\varepsilon^{-1} &= \frac{1}{2} \begin{bmatrix} \frac{\sqrt{n_1^2 n_2 + n_2 + n_1}}{n_1 \sqrt{n_2}} & \frac{\sqrt{n_1^2 n_2 + n_2 + n_1}}{\sqrt{n_2 + n_1}} \\ \frac{n_1 \sqrt{n_2}}{\sqrt{n_1^2 n_2 + n_2 + n_1}} & -\frac{\sqrt{n_1^2 n_2 + n_2 + n_1}}{\sqrt{n_2 + n_1}} \end{bmatrix}, \end{aligned} \quad (2.21)$$

and

$$B_\varepsilon = \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix},$$

we have the decomposition

$$A_\varepsilon = P_\varepsilon B_\varepsilon P_\varepsilon^{-1}. \quad (2.22)$$

Now, we are ready to prove Theorem 1.3 for the case $T_i > 0$.

Proof of Theorem 1.3 for $T_i > 0$. We prove this theorem by energy estimates. First, we note that for every $\varepsilon > 0$, (2.1) has smooth solutions in some time interval $[0, T_\varepsilon]$ dependent on ε . Let $\mathcal{C} = Op(P_\varepsilon^{-1})$, and define the norm

$$|||w(t)|||_s^2 \equiv \sum_{|\alpha| \leq s} \|\mathcal{C} \partial_x^\alpha w(t)\|^2.$$

We will bound $\partial_t |||w|||_{s'}^2$ for $\alpha \leq s'$. By a direct computation, we have

$$\begin{aligned} \partial_t \|\mathcal{C} \partial_x^\alpha w\|_{L^2}^2 &= 2\Re((\partial_t \mathcal{C}) \partial_x^\alpha w, \mathcal{C} \partial_x^\alpha w) - 2\Re(\mathcal{C}[\partial_x^\alpha, \mathcal{A}]w, \mathcal{C} \partial_x^\alpha w) \\ &\quad - 2\Re(\mathcal{C} \mathcal{A} \partial_x^\alpha w, \mathcal{C} \partial_x^\alpha w) - 2\Re(\mathcal{C} \partial_x^\alpha \mathcal{R}, \mathcal{C} \partial_x^\alpha w) \\ &=: I + II + III + IV. \end{aligned} \quad (2.23)$$

Estimate of I. Since \mathcal{C} is a bounded family of matrix-valued PsDO of order 0, it is a uniformly bounded operator from $L^2 \rightarrow L^2$. On the other hand,

$$\partial_t \mathcal{C} = Op(\partial_t P_\varepsilon^{-1}) = \sum_i \varepsilon^i \partial_{n^{(i)}} P_\varepsilon^{-1} \partial_t n^{(i)} + \varepsilon^3 \partial_{n_R^\varepsilon} P_\varepsilon^{-1} \partial_t n_R^\varepsilon.$$

From (2.1) and the expressions for n_1 and n_2 in (2.20), we have

$$\|\varepsilon \partial_t n_R^\varepsilon\|_{H^{s'-1}} \leq C(\|(n^{(i)}, u^{(i)}, \phi^{(i)})\|_{H^{\tilde{s}_i}}, \|(n_R^\varepsilon, u_R^\varepsilon)\|_{H^{s'}})$$

and since $n^{(i)}$ are the first four known profiles, we have

$$\|\partial_t n^{(i)}\|_{H^{s'-1}} \leq C(\|n^{(i)}\|_{H^{\tilde{s}_i}}, \|u^{(i)}\|_{H^{\tilde{s}_i}}, \|\phi^{(i)}\|_{H^{\tilde{s}_i}})$$

for $i = 1, 2, 3, 4$. Therefore,

$$\|\partial_t \mathcal{C}\|_{H^{s'-1}} \leq C, \quad s' > \frac{d}{2} + 1,$$

for some $C = C(\varepsilon \|n_R^\varepsilon\|_{H^{s'}}, \varepsilon \|u_R^\varepsilon\|_{H^{s'}})$. In other words, $\partial_t \mathcal{C}$ is a uniformly bounded operator from L^2 to L^2 . Consequently,

$$|I| \leq C_1 \|\partial_x^{s'} w\|^2. \quad (2.24)$$

Estimate of II in (2.23). By the definition of and $\mathcal{A}_{2,\varepsilon}$, we know that

$$[\partial_x^\alpha, \mathcal{A}_{2,\varepsilon}] = 0.$$

Since $\mathcal{A}_{1,\varepsilon}$ is a PsDO of order 1, by the commutator estimates that [15], we have

$$\|[\partial_x^\alpha, \mathcal{A}_{1,\varepsilon}]w\|_{L^2} \leq C(\|(n^{(i)}, u^{(i)})\|_{H^{\tilde{s}_i}}, \varepsilon \|(n_R^\varepsilon, u_R^\varepsilon)\|_{H^{s'}}) \|w\|_{H^\alpha},$$

so that

$$|II| \leq C(\|(n^{(i)}, u^{(i)})\|_{H^{\tilde{s}_i}}, \varepsilon \|(n_R^\varepsilon, u_R^\varepsilon)\|_{H^{s'}}) \|w\|_{H^\alpha}^2. \quad (2.25)$$

Estimate of III in (2.23). Using the diagonalization (2.22), we split

$$\begin{aligned}
& (\mathcal{CA}\partial_x^\alpha w, \mathcal{C}\partial_x^\alpha w) \\
&= (\mathcal{CA}\partial_x^\alpha w, \mathcal{C}\partial_x^\alpha w) - (Op(B_\varepsilon P_\varepsilon^{-1})\partial_x^\alpha w, \mathcal{C}\partial_x^\alpha w) \\
&\quad + (Op(B_\varepsilon P_\varepsilon^{-1})\partial_x^\alpha w, \mathcal{C}\partial_x^\alpha w) - (Op(B_\varepsilon)Op(P_\varepsilon^{-1})\partial_x^\alpha w, \mathcal{C}\partial_x^\alpha w) \\
&\quad + (Op(B_\varepsilon)Op(P_\varepsilon^{-1})\partial_x^\alpha w, \mathcal{C}\partial_x^\alpha w) \\
&= III_1 + III_2 + III_3.
\end{aligned} \tag{2.26}$$

Let us first consider the term III_1 . Since A_ε depends on $n^{(i)}$, n_R^ε in the form of $\varepsilon^i n^{(i)}$, $\varepsilon^3 n_R^\varepsilon$ for $i = 1, 2, 3, 4$, $D_{n^{(i)}} A_\varepsilon$ and $D_{n_R^\varepsilon} A_\varepsilon$ are all bounded families of symbols of order 1. Furthermore, P_ε^{-1} is a uniformly bounded family of symbols of order 0, and we have

$$\|\mathcal{CA} - Op(B_\varepsilon P_\varepsilon^{-1})\|_{L^2 \rightarrow L^2} \leq C(\|n^{(i)}\|_{H^{\tilde{s}_i}}, \varepsilon \|n_R^\varepsilon\|_{H^{s'}}).$$

Similarly, since $D_\xi^\alpha B_\varepsilon \nabla_v^\alpha P_\varepsilon^{-1}$ are bounded symbols of order $1 - \alpha$ for III_2 , we have

$$\|Op(B_\varepsilon)Op(P_\varepsilon^{-1}) - Op(B_\varepsilon P_\varepsilon^{-1})\|_{L^2 \rightarrow L^2} \leq C(\|n^{(i)}\|_{H^{\tilde{s}_i}}, \varepsilon \|n_R^\varepsilon\|_{H^{s'}}).$$

Finally, We consider III_3 . Since λ_\pm are purely imaginary when $\varepsilon < \varepsilon_2$ is sufficiently small, and $B_\varepsilon = \text{diag}[\lambda_+, \lambda_-]$ is diagonal, $B_\varepsilon^* = -B_\varepsilon$. Therefore, by using the properties of the adjoint operator (symbolic calculus), we have $B_\varepsilon^* \in S^1$ and

$$Op(B_\varepsilon)^* \sim \sum_\alpha \frac{1}{\alpha!} \partial_\xi^\alpha D_x^\alpha \bar{B}_\varepsilon(x, \xi).$$

On the other hand, since B_ε depends on $n^{(i)}$ and n_R^ε through $\varepsilon^i n^{(i)}$ and $\varepsilon^3 n_R^\varepsilon$, there exists a bounded operator \tilde{B}_ε from $L^2 \rightarrow L^2$ such that

$$\tilde{B}_\varepsilon = Op(B_\varepsilon) + Op(B_\varepsilon)^*$$

with bound

$$\|\tilde{B}_\varepsilon\|_{L^2 \rightarrow L^2} \leq C(\|n^{(i)}\|_{H^{\tilde{s}_i}}, \varepsilon \|n_R^\varepsilon\|_{H^{s'}}).$$

Consequently, we obtain from (2.26)

$$|III| \leq C(\|n^{(i)}\|_{H^{\tilde{s}_i}}, \varepsilon \|n_R^\varepsilon\|_{H^{s'}})(\|n_R^\varepsilon\|_{H^{s'}}^2 + \|u_R^\varepsilon\|_{H^{s'}}^2). \tag{2.27}$$

Estimate of IV in (2.23). Recall $\mathcal{R}(w) = \mathcal{F}_\varepsilon(w)w + \mathcal{N}(w)$ in (2.4). Since \mathcal{R} is a nonlinear bounded operator, from (2.6) we have for every $\alpha \geq 2$,

$$\|\mathcal{R}(w)\|_{H^\alpha} \leq C_\alpha(\|n_R^\varepsilon\|_{H^{s'}} + \|u_R^\varepsilon\|_{H^{s'}})$$

for some constant

$$C_\alpha = C_\alpha(\|(n^{(i)}, u^{(i)}, \phi^{(i)})\|_{H^{\tilde{s}_i}}, \varepsilon \|n_R^\varepsilon\|_{H^{s'}}, \varepsilon \|u_R^\varepsilon\|_{H^{s'}}).$$

Since $\mathcal{C} \in S^0$ uniformly in ε , we obtain

$$\|IV\|_{s'} \leq C(1 + \|n_R^\varepsilon\|_{H^{s'}}^2 + \|u_R^\varepsilon\|_{H^{s'}}^2). \tag{2.28}$$

Therefore, from (2.23), (2.25), (2.27) and (2.28) we obtain

$$\partial_t \|w\|_{s'}^2 \leq C(\|n^{(i)}\|_{H^{\tilde{s}_i}}, \varepsilon \|n_R^\varepsilon\|_{H^{s'}})(1 + \|w\|_{H^{s'}}^2).$$

We claim that $\|\cdot\|_{H^{s'}}$ and $|||\cdot|||_{s'}$ are equivalent. Since \mathcal{C} is a bounded family symbols of S^0 , we have

$$\|\mathcal{C}\partial_x^\alpha w\|_{L^2}^2 \leq C(\|n^{(i)}\|_{H^{\tilde{s}_i}}, \varepsilon\|n_R^\varepsilon\|_{H^{s'}})\|\partial_x^\alpha w\|_{L^2}^2, \quad \alpha \leq s'$$

and hence

$$|||w|||_{s'}^2 \leq C(\|n^{(i)}\|_{H^{\tilde{s}_i}}, \varepsilon\|n_R^\varepsilon\|_{H^{s'}})\|w\|_{H^{s'}}^2.$$

On the other hand, since P_ε^{-1} and P_ε depend on ξ , $n^{(i)}$, n_R^ε through $\sqrt{\varepsilon}\xi$, $\varepsilon^i n^{(i)}$, $\varepsilon^3 n_R^\varepsilon$, we therefore have

$$Op(P_\varepsilon)Op(P_\varepsilon^{-1}) = I + \varepsilon^{3/2}\mathcal{P},$$

for some $L^2 \rightarrow L^2$ bounded operator \mathcal{P} . Hence,

$$\|\partial_x^\alpha w\|_{L^2}^2 \leq \|Op(P_\varepsilon)Op(P_\varepsilon^{-1})\partial_x^\alpha w\|_{L^2}^2 + \varepsilon^3 C(\|n^{(i)}\|_{H^{\tilde{s}_i}}, \varepsilon\|n_R^\varepsilon\|_{H^{s'}})\|\partial_x^\alpha w\|_{L^2}^2.$$

By the L^2 -boundedness of $Op(P_\varepsilon)$, when ε is sufficiently small we have

$$\|\partial_x^\alpha w\|_{L^2}^2 \leq 2C(\|n^{(i)}\|_{H^{\tilde{s}_i}}, \varepsilon\|n_R^\varepsilon\|_{H^{s'}})\|Op(P_\varepsilon^{-1})\partial_x^\alpha w\|_{L^2}^2.$$

Summation over $|\alpha| \leq s'$ yields the equivalence between $\|\cdot\|_{H^{s'}}$ and $|||\cdot|||_{s'}$.

Therefore, we finally obtain the estimate of the form

$$\partial_t |||w|||_{s'}^2 \leq C(\|n^{(i)}\|_{H^{s'}}, \varepsilon\|n_R^\varepsilon\|_{H^{s'}})(1 + |||w|||_{s'}^2).$$

Since C depends on $\|n_R^\varepsilon\|_{H^{s'}}$ through $\varepsilon\|n_R^\varepsilon\|_{H^{s'}}$, we obtain an existence time $T_\varepsilon \geq \tau$ for any $\tau > 0$ uniformly in ε . From the decomposition of ϕ_R^ε in (2.7), we recover the uniform in ε estimate for $\|\phi_R^\varepsilon\|_{H^{s'}}$.

The proof of Theorem 1.3 for the case $T_i > 0$ is then complete for $s' = 2$. We indeed have proved a stronger result that holds for any $s' \geq 2$ integers. \square

3. UNIFORM ENERGY ESTIMATES: THE CASE $T_i = 0$

In the cold plasma ($T_i = 0$) case, the procedure in Section 2 is not applicable for two main reasons: the system cannot be symmetrized and P_ε^{-1} is not a PsDO of order 0. In this section, we handle this case, which requires a combination of energy method and analysis of remainder equation (1.17).

Throughout this section, we set $T_i = 0$ and renormalize all the other constants to be 1. Hence $V = 1$, and from (1.17) we obtain the following remainder equation

$$\begin{cases} \partial_t n_R^\varepsilon - \frac{1-u}{\varepsilon} \partial_x n_R^\varepsilon + \frac{n}{\varepsilon} \partial_x u_R^\varepsilon + \partial_x \tilde{n} u_R^\varepsilon + \partial_x \tilde{u} n_R^\varepsilon + \varepsilon \mathcal{R}_1 = 0 & (3.1a) \end{cases}$$

$$\begin{cases} \partial_t u_R^\varepsilon - \frac{1-u}{\varepsilon} \partial_x u_R^\varepsilon + \partial_x \tilde{u} u_R^\varepsilon + \varepsilon \mathcal{R}_2 = -\frac{1}{\varepsilon} \partial_x \phi_R^\varepsilon & (3.1b) \end{cases}$$

$$\begin{cases} \varepsilon \partial_x^2 \phi_R^\varepsilon = (\phi_R^\varepsilon + \varepsilon \phi^{(1)} \phi_R^\varepsilon - n_R^\varepsilon) + \varepsilon^2 \mathcal{R}_3, & (3.1c) \end{cases}$$

where $\mathcal{R}_1, \mathcal{R}_2$ and \mathcal{R}_3 are given by (1.18) with $T_i = 0$. In particular, \mathcal{R}_1 and \mathcal{R}_2 depend only on $(n^{(i)}, u^{(i)})$ and \mathcal{R}_3 does not involve any derivatives of ϕ_R^ε .

In the following, we will give uniform estimates of system (3.1). To simplify the proof slightly, we will assume that (3.1) has smooth solutions in very small time $\tau_\varepsilon > 0$ dependent on $\varepsilon > 0$. Recall that

$$|||(u_R^\varepsilon, \phi_R^\varepsilon)|||_\varepsilon^2 = \|u_R^\varepsilon\|_{H^2}^2 + \|\phi_R^\varepsilon\|_{H^2}^2 + \varepsilon \|\partial_x^3 u_R^\varepsilon\|^2 + \varepsilon \|\partial_x^3 \phi_R^\varepsilon\|^2 + \varepsilon^2 \|\partial_x^4 \phi_R^\varepsilon\|^2. \quad (3.2)$$

Let \tilde{C} be a constant independent of ε , which will be determined later, much larger than the bound $\|(u_R^\varepsilon, \phi_R^\varepsilon)(0)\|_\varepsilon^2$ of the initial data. It is classical that there exists $\tau_\varepsilon > 0$ such that on $[0, \tau_\varepsilon]$,

$$\|n_R^\varepsilon\|_{H^2}^2, \quad \|(u_R^\varepsilon, \phi_R^\varepsilon)(t)\|_\varepsilon^2 \leq \tilde{C}.$$

As a direct corollary, there exists some $\varepsilon_1 > 0$ such that n is bounded from above and below $1/2 < n < 3/2$ and u is bounded by $|u| < 1/2$ when $\varepsilon < \varepsilon_1$. Since \mathcal{R}_3 is a smooth function of ϕ_R^ε (see Appendix), there exists some constant $C_1 = C_1(\varepsilon\tilde{C})$ for any $\alpha, \beta \geq 0$ such that

$$|\partial_{\phi^{(i)}}^\alpha \partial_{\phi_R^\varepsilon}^\beta \mathcal{R}_3| \leq C_1 = C_1(\varepsilon\tilde{C}), \quad (3.3)$$

where $C_1(\cdot)$ can be chose to be nondecreasing in its argument.

We will show that for any given $\tau > 0$, there is some $\varepsilon_0 > 0$, such that the existence time $\tau_\varepsilon > \tau$ for any $0 < \varepsilon < \varepsilon_0$. We first prove the following Lemma 3.1-3.3, in which we bound n_R^ε and $\partial_t \phi_R^\varepsilon$ in terms of ϕ_R^ε .

Lemma 3.1. *Let $(n_R^\varepsilon, u_R^\varepsilon, \phi_R^\varepsilon)$ be a solution to (3.1) and $\alpha \geq 0$ be an integer. There exist some constants $0 < \varepsilon_1 < 1$ and $C_1 = C_1(\varepsilon\tilde{C})$ such that for every $0 < \varepsilon < \varepsilon_1$,*

$$C_1^{-1} \|\partial_x^\alpha n_R^\varepsilon\|^2 \leq \|\partial_x^\alpha \phi_R^\varepsilon\|^2 + \varepsilon \|\partial_x^{\alpha+1} \phi_R^\varepsilon\|^2 + \varepsilon^2 \|\partial_x^{\alpha+2} \phi_R^\varepsilon\|^2 \leq C_1 \|\partial_x^\alpha n_R^\varepsilon\|^2. \quad (3.4)$$

Proof. When $\alpha = 0$, taking inner product of (3.1c) with ϕ_R^ε , we have

$$\|\phi_R^\varepsilon\|^2 + \varepsilon \|\partial_x \phi_R^\varepsilon\|^2 = \int n_R^\varepsilon \phi_R^\varepsilon - \int \varepsilon \phi^{(1)} |\phi_R^\varepsilon|^2 - \int \varepsilon^2 \mathcal{R}_3 \phi_R^\varepsilon. \quad (3.5)$$

From (A.5) in the Appendix, we have

$$\|\mathcal{R}_3\|_{L^2} \leq C_1(\varepsilon\tilde{C}) \|\phi_R^\varepsilon\|.$$

When $\varepsilon < \varepsilon_1$ is sufficiently small, $C_1(\varepsilon\tilde{C}) \leq C_1(1)$ is a fixed constant, and therefore

$$\left| \int \varepsilon^2 \mathcal{R}_3 \phi_R^\varepsilon \right| \leq \frac{1}{8} \|\phi_R^\varepsilon\|^2. \quad (3.6)$$

Since $\phi^{(1)}$ is known and is bounded in L^∞ , there exists some $0 < \varepsilon_1 < 1$ such that for $0 < \varepsilon < \varepsilon_1$,

$$\left| \int \varepsilon \phi^{(1)} |\phi_R^\varepsilon|^2 \right| \leq \frac{1}{8} \|\phi_R^\varepsilon\|^2. \quad (3.7)$$

By applying the Hölder's inequality to the first term on the RHS of (3.5), we have

$$\left| \int n_R^\varepsilon \phi_R^\varepsilon \right| \leq \frac{1}{4} \|\phi_R^\varepsilon\|^2 + \|n_R^\varepsilon\|^2. \quad (3.8)$$

By (3.5)-(3.8),

$$\|\phi_R^\varepsilon\|^2 + \varepsilon \|\partial_x \phi_R^\varepsilon\|^2 \leq \frac{1}{2} \|\phi_R^\varepsilon\|^2 + \|n_R^\varepsilon\|^2.$$

Hence, we have shown that there exists some $\varepsilon_1 > 0$ such that for $0 < \varepsilon < \varepsilon_1$,

$$\|\phi_R^\varepsilon\|^2 + \varepsilon \|\partial_x \phi_R^\varepsilon\|^2 \leq 2 \|n_R^\varepsilon\|^2. \quad (3.9)$$

Taking inner product with $\varepsilon \partial_x^2 \phi_R^\varepsilon$ and integration by parts, we have similarly

$$\varepsilon \|\partial_x \phi_R^\varepsilon\|^2 + \varepsilon^2 \|\partial_x^2 \phi_R^\varepsilon\|^2 \leq 2 \|n_R^\varepsilon\|^2. \quad (3.10)$$

On the other hand, from the equation (3.1c), there exist some C such that

$$\begin{aligned} \|n_R^\varepsilon\|^2 &\leq \|\phi_R^\varepsilon\|^2 + \varepsilon^2 \|\partial_x^2 \phi_R^\varepsilon\|^2 + C\varepsilon^2 \|\phi_R^\varepsilon\|^2 + (C_1(1))^2 \|\phi_R^\varepsilon\|^2 \\ &\leq C(\|\phi_R^\varepsilon\|^2 + \varepsilon^2 \|\partial_x^2 \phi_R^\varepsilon\|^2). \end{aligned} \quad (3.11)$$

Putting (3.9)-(3.11) together, we deduce the inequality (3.4) for $\alpha = 0$.

For higher order inequalities, we differentiate the Poisson equation (3.1c) with ∂_x^α and then take inner product with $\partial_x^\alpha \phi_R^\varepsilon$ and $\varepsilon \partial_x^{\alpha+2} \phi_R^\varepsilon$ separately. The Lemma follows by the same procedure of the case $\alpha = 0$. \square

Recall $\|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon$ in (3.2). We remark that only $\|n_R^\varepsilon\|_{H^2}$ can be bounded in terms of $\|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon$ and no higher order derivatives of n_R^ε is allowed in Lemma 3.1. This is one of the reasons that why the estimate in the section is delicate.

Lemma 3.2. *Let $(n_R^\varepsilon, u_R^\varepsilon, \phi_R^\varepsilon)$ be a solution to (3.1). There exist some constant C and $C_1 = C_1(\varepsilon \tilde{C})$, such that*

$$\|\varepsilon \partial_t n_R^\varepsilon\|^2 \leq C(\|\phi_R^\varepsilon\|_{H^1}^2 + \|u_R^\varepsilon\|_{H^1}^2 + \varepsilon \|\partial_x^2 \phi_R^\varepsilon\|^2 + \varepsilon^2 \|\partial_x^3 \phi_R^\varepsilon\|^2) + C\varepsilon. \quad (3.12)$$

$$\|\varepsilon \partial_{tx} n_R^\varepsilon\|^2 \leq C_1(\|u_R^\varepsilon\|_{H^2}^2 + \|\phi_R^\varepsilon\|_{H^2}^2 + \varepsilon \|\partial_x^3 \phi_R^\varepsilon\|^2 + \varepsilon^2 \|\partial_x^4 \phi_R^\varepsilon\|^2) + C\varepsilon. \quad (3.13)$$

By (3.2), it is useful to rewrite in the form

$$\|\varepsilon \partial_t n_R^\varepsilon\|_{H^1}^2 \leq C_1 \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2 + C\varepsilon.$$

Proof. From (3.1a), we have

$$\varepsilon \partial_t n_R^\varepsilon = (1-u) \partial_x n_R^\varepsilon - n \partial_x u_R^\varepsilon - \varepsilon \partial_x \tilde{u} n_R^\varepsilon - \varepsilon \partial_x \tilde{n} u_R^\varepsilon - \varepsilon^2 \mathcal{R}_1.$$

Since $1/2 < n < 3/2$ and $|u| < 1/2$, taking L^2 -norm yields

$$\begin{aligned} \|\varepsilon \partial_t n_R^\varepsilon\|^2 &\leq \|(1-u) \partial_x n_R^\varepsilon\|^2 + \|n \partial_x u_R^\varepsilon\|^2 + \varepsilon^2 \|\partial_x \tilde{u} n_R^\varepsilon\|^2 + \varepsilon^2 \|\partial_x \tilde{n} u_R^\varepsilon\|^2 + \varepsilon^4 \|\mathcal{R}_1\|^2 \\ &\leq C(\|\partial_x n_R^\varepsilon\|^2 + \|\partial_x u_R^\varepsilon\|^2) + C\varepsilon^2(\varepsilon^2 + \|n_R^\varepsilon\|^2 + \|u_R^\varepsilon\|^2). \end{aligned}$$

Applying (3.4) with $\alpha = 1$, we deduce (3.12).

To prove (3.13), we take ∂_x of (3.1a) to obtain

$$\|\varepsilon \partial_{tx} n_R^\varepsilon\|^2 \leq C(\|u_R^\varepsilon\|_{H^2}^2 + \|n_R^\varepsilon\|_{H^2}^2) + C\varepsilon^6 \int |\partial_x u_R^\varepsilon|^2 |\partial_x n_R^\varepsilon|^2 + C\varepsilon^4.$$

We note that

$$C\varepsilon^6 \|\partial_x u_R^\varepsilon\|_{L^\infty}^2 \|\partial_x n_R^\varepsilon\|^2 \leq C\varepsilon^6 \|u_R^\varepsilon\|_{H^2}^2 \|n_R^\varepsilon\|_{H^1}^2 \leq C(\varepsilon \tilde{C}) \|u_R^\varepsilon\|_{H^2}^2.$$

The Lemma then follows from Lemma 3.1. \square

Lemma 3.3. *Let $(n_R^\varepsilon, u_R^\varepsilon, \phi_R^\varepsilon)$ be a solution to (3.1) and $\alpha \geq 0$ be an integer. There exist some constant $C_1 = C(\varepsilon \tilde{C})$ and $\varepsilon_1 > 0$ such that for any $0 < \varepsilon < \varepsilon_1$,*

$$\varepsilon \|\partial_t \partial_x^{\alpha+1} \phi_R^\varepsilon\|^2 + \|\partial_t \partial_x^\alpha \phi_R^\varepsilon\|^2 \leq 2 \|\partial_t \partial_x^\alpha n_R^\varepsilon\|^2 + C_1.$$

Proof. The proof is similar to that of Lemma 3.1. When $\alpha = 0$, by first taking ∂_t of (3.1c) and then taking inner product with $\partial_t \phi_R^\varepsilon$, we have

$$\begin{aligned} \varepsilon \|\partial_{tx} \phi_R^\varepsilon\|^2 + \|\partial_t \phi_R^\varepsilon\|^2 &= \int \partial_t n_R^\varepsilon \partial_t \phi_R^\varepsilon - \int (\varepsilon \partial_t (\phi^{(1)} \phi_R^\varepsilon) + \varepsilon^2 \partial_t \mathcal{R}_3) \partial_t \phi_R^\varepsilon \\ &\leq \frac{1}{4} \|\partial_t \phi_R^\varepsilon\|^2 + \|\partial_t n_R^\varepsilon\|^2 + C(\varepsilon \tilde{C}) \varepsilon (\|\phi_R^\varepsilon\|^2 + \|\partial_t \phi_R^\varepsilon\|^2), \end{aligned}$$

thanks to (A.6) in Lemma A.1 in Appendix. Therefore, there exists some $\varepsilon_1 > 0$ such that when $\varepsilon < \varepsilon_1$,

$$\varepsilon \|\partial_{tx} \phi_R^\varepsilon\|^2 + \|\partial_t \phi_R^\varepsilon\|^2 \leq 2 \|\partial_t n_R^\varepsilon\|^2 + C(\varepsilon \tilde{C}).$$

When $\alpha = 1$, we take ∂_{tx} of (3.1c) and then take inner product with $\partial_{tx} \phi_R^\varepsilon$ to obtain

$$\varepsilon \|\partial_t \partial_x^2 \phi_R^\varepsilon\|^2 + \|\partial_{tx} \phi_R^\varepsilon\|^2 \leq 2 \|\partial_{tx} n_R^\varepsilon\|^2 + C(\varepsilon \tilde{C}).$$

The case of $\alpha \geq 2$ can be proved similarly. \square

The rest of this section is devoted to the proof of Theorem 1.3 for the case $T_i = 0$, which is divided into the following several subsections.

3.1. Zeroth, first and second order estimates.

Proposition 3.1. *Let $(n_R^\varepsilon, u_R^\varepsilon, \phi_R^\varepsilon)$ be a solution to (3.1) and $\gamma = 0, 1, 2$, then*

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\partial_x^\gamma u_R^\varepsilon\|^2 + \frac{1}{2} \frac{d}{dt} \left[\int \frac{1 + \varepsilon \phi^{(1)}}{n} |\partial_x^\gamma \phi_R^\varepsilon|^2 + \int \frac{\varepsilon}{n} |\partial_x^{\gamma+1} \phi_R^\varepsilon|^2 \right] \\ \leq C_1 (1 + \varepsilon^2 \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2) (1 + \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2). \end{aligned} \quad (3.14)$$

Proof. We take ∂_x^γ of (3.1b) and then take inner product of $\partial_x^\gamma u_R^\varepsilon$. Integrating by parts, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\partial_x^\gamma u_R^\varepsilon\|^2 - \frac{1}{\varepsilon} \int \partial_x^{\gamma+1} u_R^\varepsilon \partial_x^\gamma u_R^\varepsilon + \int \partial_x^\gamma [(\tilde{u} + \varepsilon^2 u_R^\varepsilon) \partial_x u_R^\varepsilon] \partial_x^\gamma u_R^\varepsilon \\ + \int \partial_x^\gamma [\partial_x \tilde{u} u_R^\varepsilon] \partial_x^\gamma u_R^\varepsilon + \int \partial_x^\gamma [\varepsilon \mathcal{R}_2] \partial_x^\gamma u_R^\varepsilon \\ = \int \partial_x^\gamma \phi_R^\varepsilon \frac{\partial_x^{\gamma+1} u_R^\varepsilon}{\varepsilon} =: I^{(\gamma)}. \end{aligned} \quad (3.15)$$

Estimate of the LHS of (3.15). The second term on the LHS of (3.15) vanishes by integration by parts. The third term on the LHS of (3.15) consists of two parts. For the first part, for $0 \leq \gamma \leq 2$, we have

$$\begin{aligned} \int \partial_x^\gamma (\tilde{u} \partial_x u_R^\varepsilon) \partial_x^\gamma u_R^\varepsilon &= \int \tilde{u} \partial_x^{\gamma+1} u_R^\varepsilon \partial_x^\gamma u_R^\varepsilon + \sum_{0 \leq \beta \leq \gamma-1} C_\gamma^\beta \int \partial_x^{\gamma-\beta} \tilde{u} \partial_x^{\beta+1} u_R^\varepsilon \partial_x^\gamma u_R^\varepsilon \\ &= -\frac{1}{2} \int \partial_x \tilde{u} \partial_x^\gamma u_R^\varepsilon \partial_x^\gamma u_R^\varepsilon + \sum_{0 \leq \beta \leq \gamma-1} C_\gamma^\beta \int \partial_x^{\gamma-\beta} \tilde{u} \partial_x^{\beta+1} u_R^\varepsilon \partial_x^\gamma u_R^\varepsilon \\ &\leq C \|u_R^\varepsilon\|_{H^2}^2, \end{aligned}$$

where, when $\gamma = 0$, there is no such “summation” term. For the second part, after integration by parts, we have for $0 \leq \gamma \leq 2$

$$\begin{aligned}
& \varepsilon^2 \int \partial_x^\gamma (u_R^\varepsilon \partial_x u_R^\varepsilon) \partial_x^\gamma u_R^\varepsilon \\
&= -\frac{\varepsilon^2}{2} \int \partial_x u_R^\varepsilon \partial_x^\gamma u_R^\varepsilon \partial_x^\gamma u_R^\varepsilon + \sum_{0 \leq \beta \leq \gamma-1} C_\gamma^\beta \varepsilon^2 \int \partial_x^{\gamma-\beta} u_R^\varepsilon \partial_x^{\beta+1} u_R^\varepsilon \partial_x^\gamma u_R^\varepsilon \\
&\leq C \varepsilon^2 \|\partial_x u_R^\varepsilon\|_{L^\infty} \|u_R^\varepsilon\|_{H^\gamma}^2 \\
&\leq C(\varepsilon^2 \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon) \|u_R^\varepsilon\|_{H^\gamma}^2,
\end{aligned}$$

where $\|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon$ is given in (3.2). For the last two terms on the LHS of (3.15), since $\partial_x^\gamma \mathcal{R}_2$ is integrable by (1.18c) and Theorem 1.2, they can be similarly bounded by $\|u_R^\varepsilon\|_{H^\gamma}^2 + C\varepsilon^2$. In summary, the last four terms on the LHS of (3.15) are bounded by

$$C(1 + \varepsilon^2 \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon)(1 + \|u_R^\varepsilon\|_{H^\gamma}^2). \quad (3.16)$$

Estimate of the RHS term $I^{(\gamma)}$ in (3.15). Taking ∂_x^γ of (3.1a), we have

$$\begin{aligned}
\frac{\partial_x^{\gamma+1} u_R^\varepsilon}{\varepsilon} &= \frac{1}{n} \left[\frac{(1-u)}{\varepsilon} \partial_x^{\gamma+1} n_R^\varepsilon - \partial_t \partial_x^\gamma n_R^\varepsilon - \sum_{0 \leq \beta \leq \gamma-1} C_\gamma^\beta \partial_x^{\gamma-\beta} (\tilde{n} + \varepsilon^2 n_R^\varepsilon) \partial_x^{\beta+1} u_R^\varepsilon \right. \\
&\quad - \sum_{0 \leq \beta \leq \gamma-1} C_\gamma^\beta \partial_x^{\gamma-\beta} (\tilde{u} + \varepsilon^2 u_R^\varepsilon) \partial_x^{\beta+1} n_R^\varepsilon - \sum_{0 \leq \beta \leq \gamma} C_\gamma^\beta \partial_x^\beta u_R^\varepsilon \partial_x^{\gamma-\beta+1} \tilde{n} \\
&\quad \left. - \sum_{0 \leq \beta \leq \gamma} C_\gamma^\beta \partial_x^\beta n_R^\varepsilon \partial_x^{\gamma-\beta+1} \tilde{u} - \varepsilon \partial_x^\gamma \mathcal{R}_1 \right] =: \sum_{i=1}^7 A_i^{(\gamma)}. \quad (3.17)
\end{aligned}$$

Accordingly, $I^{(\gamma)}$ is decomposed into

$$I^{(\gamma)} = \sum_{i=1}^7 I_i^{(\gamma)} = \sum_{i=1}^7 \int \partial_x^2 \phi_R^\varepsilon A_i^{(\gamma)}. \quad (3.18)$$

We first estimate the terms $I_i^{(\gamma)}$ for $3 \leq i \leq 7$ and leave $I_1^{(\gamma)}$ and $I_2^{(\gamma)}$ in the following two lemmas.

Estimate of $I_3^{(\gamma)}$ in (3.18). We divide it into two parts

$$\begin{aligned}
I_3^{(\gamma)} &= \sum_{0 \leq \beta \leq \gamma-1} C_\gamma^\beta \int \partial_x^2 \phi_R^\varepsilon \partial_x^{\gamma-\beta} \tilde{n} \partial_x^{\beta+1} u_R^\varepsilon + \sum_{0 \leq \beta \leq \gamma-1} C_\gamma^\beta \varepsilon^2 \int \partial_x^2 \phi_R^\varepsilon \partial_x^{\gamma-\beta} n_R^\varepsilon \partial_x^{\beta+1} u_R^\varepsilon \\
&=: I_{31}^{(\gamma)} + I_{32}^{(\gamma)}.
\end{aligned}$$

The first one is easily bounded by

$$I_{31}^{(\gamma)} \leq C(\|u_R^\varepsilon\|_{H^2}^2 + \|\phi_R^\varepsilon\|_{H^2}^2).$$

For the second term $I_{32}^{(\gamma)}$, since the order of the derivative on n_R^ε does not exceed 2, using Hölder inequality, Sobolev embedding $H^1 \hookrightarrow L^\infty$ and then Lemma 3.1, we

deduce

$$\begin{aligned} I_{32}^{(\gamma)} &\leq C\varepsilon^2 \|\partial_x^2 \phi_R^\varepsilon\|_{L^\infty} \|n_R^\varepsilon\|_{H^2} \|u_R^\varepsilon\|_{H^2} \\ &\leq C_1(1 + \varepsilon^2 \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2) (\|u_R^\varepsilon\|_{H^2}^2 + \varepsilon \|\phi_R^\varepsilon\|_{H^3}^2), \end{aligned}$$

where $\|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon$ is given in (3.2). Hence

$$I_3^{(\gamma)} \leq C_1(1 + \varepsilon^2 \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2) \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2.$$

Estimate of $I_4^{(\gamma)}$ in (3.18). The term $I_4^{(\gamma)}$ is bounded similarly

$$I_4^{(\gamma)} \leq C_1(1 + \varepsilon^2 \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2) \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2.$$

Estimate of $I_5^{(\gamma)}, I_6^{(\gamma)}, I_7^{(\gamma)}$ in (3.18). Since the terms $I_i^{(\gamma)}$ for $i = 5, 6, 7$ are bilinear or linear in the unknowns, they can be bounded by

$$I_{5,6,7}^{(\gamma)} \leq C_1(1 + \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2).$$

In summary, we have

$$\sum_{i=3}^7 I_i^{(\gamma)} \leq C_1(1 + \varepsilon^2 \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2)(1 + \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2).$$

We deduce Proposition 3.1 by the following Lemma 3.4 and 3.5. \square

Lemma 3.4 (*Estimate of $I_1^{(\gamma)}$*). *Let $(n_R^\varepsilon, u_R^\varepsilon, \phi_R^\varepsilon)$ be a solution to (3.1), we have*

$$I_1^{(\gamma)} \leq C_1(1 + \varepsilon^2 \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2)(1 + \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2),$$

where $I_1^{(\gamma)}$ is defined in (3.18) and $\|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon$ is given in (3.2).

Proof of Lemma 3.4. Taking $\partial_x^{\gamma+1}$ of (3.1c), we have

$$\partial_x^{\gamma+1} n_R^\varepsilon = \partial_x^{\gamma+1} \phi_R^\varepsilon - \varepsilon \partial_x^{\gamma+3} \phi_R^\varepsilon + \varepsilon \partial_x^{\gamma+1} (\phi^{(1)} \phi_R^\varepsilon) + \varepsilon^2 \partial_x^{\gamma+1} \mathcal{R}_3 =: \sum_{i=1}^4 B_i^{(\gamma)}.$$

Accordingly, $I_1^{(\gamma)}$ is decomposed into

$$I_1^{(\gamma)} = \sum_{i=1}^4 \int \partial_x^\gamma \phi_R^\varepsilon \left[\frac{(1-u)}{\varepsilon n} B_i^{(\gamma)} \right] =: \sum_{i=1}^4 I_{1i}^{(\gamma)}.$$

Estimate of $I_{11}^{(\gamma)}$. Integrating by parts yields

$$\begin{aligned} I_{11}^{(\gamma)} &= \int \frac{(1-u)}{\varepsilon n} \partial_x^\gamma \phi_R^\varepsilon \partial_x^{\gamma+1} \phi_R^\varepsilon \\ &= -\frac{1}{2} \int \partial_x \left[\frac{(1-u)}{\varepsilon n} \right] |\partial_x^\gamma \phi_R^\varepsilon|^2 \\ &\leq C \|\partial_x^\gamma \phi_R^\varepsilon\|^2 + C\varepsilon^2 (\|\partial_x u_R^\varepsilon\|_{L^\infty} + \|\partial_x n_R^\varepsilon\|_{L^\infty}) \|\partial_x^\gamma \phi_R^\varepsilon\|^2, \end{aligned}$$

thanks to the fact

$$\partial_x \left[\frac{(1-u)}{\varepsilon n} \right] \leq C(|\partial_x \tilde{n}| + |\partial_x \tilde{u}|) + \varepsilon^2 (|\partial_x n_R^\varepsilon| + |\partial_x u_R^\varepsilon|).$$

Using Sobolev embedding and Lemma 3.1, by (3.2) we have

$$I_{11}^{(\gamma)} \leq C \|\partial_x^\gamma \phi_R^\varepsilon\|^2 + C_1(1 + \varepsilon \|(u_R^\varepsilon, u_R^\varepsilon)\|_\varepsilon) \|\partial_x^\gamma \phi_R^\varepsilon\|^2. \quad (3.19)$$

Estimate of $I_{12}^{(\gamma)}$. By integration by parts twice, we have

$$\begin{aligned} I_{12}^{(\gamma)} &= - \int \partial_x^\gamma \phi_R^\varepsilon \left[\frac{(1-u)}{n} \partial_x^{\gamma+3} \phi_R^\varepsilon \right] \\ &= - \frac{3}{2} \int \partial_x \left[\frac{(1-u)}{n} \right] |\partial_x^{\gamma+1} \phi_R^\varepsilon|^2 - \int \partial_x^2 \left[\frac{(1-u)}{n} \right] \partial_x^\gamma \phi_R^\varepsilon \partial_x^{\gamma+1} \phi_R^\varepsilon \\ &=: I_{121}^{(\gamma)} + I_{122}^{(\gamma)}. \end{aligned} \quad (3.20)$$

Note that

$$\partial_x \left[\frac{(V-u)}{n} \right] \leq C \left(\varepsilon (|\partial_x \tilde{n}| + |\partial_x \tilde{u}|) + \varepsilon^3 (|\partial_x n_R^\varepsilon| + |\partial_x u_R^\varepsilon|) \right).$$

Similar to the bound for $I_{11}^{(\gamma)}$ in (3.19), we have

$$I_{121}^{(\gamma)} \leq C \varepsilon \|\partial_x^{\gamma+1} \phi_R^\varepsilon\|^2 + C_1(1 + \varepsilon \|(u_R^\varepsilon, u_R^\varepsilon)\|_\varepsilon) (\varepsilon \|\partial_x^{\gamma+1} \phi_R^\varepsilon\|^2). \quad (3.21)$$

Note that

$$\begin{aligned} \left| \partial_x^2 \left[\frac{(1-u)}{n} \right] \right| &\leq C \left(\varepsilon + \varepsilon^3 (|\partial_x^2 n_R^\varepsilon| + |\partial_x^2 u_R^\varepsilon|) \right. \\ &\quad \left. + \varepsilon^4 (|\partial_x n_R^\varepsilon| + |\partial_x u_R^\varepsilon|) + \varepsilon^6 (|\partial_x n_R^\varepsilon|^2 + |\partial_x u_R^\varepsilon|^2) \right). \end{aligned}$$

By Hölder inequality, Sobolev embedding and Lemma 3.1 for $0 \leq \gamma \leq 2$, we obtain

$$\begin{aligned} I_{122}^{(\gamma)} &\leq C \|\partial_x^\gamma \phi_R^\varepsilon\|_{L^\infty} \|\partial_x^2 \left[\frac{(V-u)}{n} \right]\| \|\partial_x^{\gamma+1} \phi_R^\varepsilon\| \\ &\leq C \varepsilon \|\partial_x^\gamma \phi_R^\varepsilon\|_{H^1} (1 + \varepsilon^2 (\|n_R^\varepsilon\|_{H^2}^2 + \|u_R^\varepsilon\|_{H^2}^2)) \|\partial_x^{\gamma+1} \phi_R^\varepsilon\| \\ &\leq C_1(1 + \varepsilon^2 \|(u_R^\varepsilon, u_R^\varepsilon)\|_\varepsilon^2) (\varepsilon \|\phi_R^\varepsilon\|_{H^{\gamma+1}}^2). \end{aligned} \quad (3.22)$$

Therefore, combining (3.20), (3.21) and (3.22), we obtain

$$I_{12}^{(\gamma)} \leq C_1(1 + \varepsilon^2 \|(u_R^\varepsilon, u_R^\varepsilon)\|_\varepsilon^2) (\varepsilon \|\phi_R^\varepsilon\|_{H^{\gamma+1}}^2).$$

Estimate for $I_{13}^{(\gamma)}$. The estimate for $I_{13}^{(\gamma)}$ is similar to that for $I_{11}^{(\gamma)}$ in (3.19),

$$I_{13}^{(\gamma)} \leq C_1(1 + \varepsilon \|(u_R^\varepsilon, u_R^\varepsilon)\|_\varepsilon) \|\phi_R^\varepsilon\|_{H^\gamma}^2.$$

Estimate of $I_{14}^{(\gamma)}$. By integration by parts and Lemma A.1, we deduce

$$\begin{aligned} I_{14}^{(2)} &= \varepsilon \int \partial_x^\gamma \phi_R^\varepsilon \left[\frac{(1-u)}{n} \partial_x^{\gamma+1} \mathcal{R}_3 \right] \\ &= - \varepsilon \int \partial_x \left[\frac{(1-u)}{n} \right] \partial_x^\gamma \phi_R^\varepsilon \partial_x^{\gamma+1} \mathcal{R}_3 - \varepsilon \int \frac{(1-u)}{n} \partial_x^{\gamma+1} \phi_R^\varepsilon \partial_x^\gamma \mathcal{R}_3 \\ &\leq C_1 \varepsilon^2 (1 + \|\partial_x n_R^\varepsilon\|_{L^\infty} + \|\partial_x u_R^\varepsilon\|_{L^\infty}) \|\phi_R^\varepsilon\|_{H^\gamma}^2 + C_1 \varepsilon \|\phi_R^\varepsilon\|_{H^{\gamma+1}} \|\phi_R^\varepsilon\|_{H^\gamma} \\ &\leq C_1(1 + \varepsilon \|(u_R^\varepsilon, u_R^\varepsilon)\|_\varepsilon) (\varepsilon \|\phi_R^\varepsilon\|_{H^{\gamma+1}}^2). \end{aligned}$$

The proof of Lemma 3.4 is then complete. \square

Lemma 3.5 (*Estimate of $I_2^{(\gamma)}$*). Let $(n_R^\varepsilon, u_R^\varepsilon, \phi_R^\varepsilon)$ be a solution to (3.1) and $0 \leq \gamma \leq 2$. The following inequality holds

$$I_2^{(\gamma)} \leq -\frac{1}{2} \frac{d}{dt} \int \frac{1 + \varepsilon \phi^{(1)}}{n} |\partial_x^\gamma \phi_R^\varepsilon|^2 dx - \frac{1}{2} \frac{d}{dt} \int \frac{\varepsilon}{n} |\partial_x^{\gamma+1} \phi_R^\varepsilon|^2 dx \\ + C_1(1 + \varepsilon^2 \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2)(1 + \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2),$$

where $I_2^{(\gamma)}$ is defined in (3.18) and $\|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon$ is given in (3.2).

Proof of Lemma 3.5. Taking $\partial_x \partial_x^\gamma$ of (3.1c), we have

$$\partial_t \partial_x^\gamma n_R^\varepsilon = \partial_t \partial_x^\gamma \phi_R^\varepsilon - \varepsilon \partial_t \partial_x^{\gamma+2} \phi_R^\varepsilon + \varepsilon \partial_t \partial_x^\gamma (\phi^{(1)} \phi_R^\varepsilon) + \varepsilon^2 \partial_t \partial_x^\gamma \mathcal{R}_3 =: \sum_i D_i^{(2)}.$$

Accordingly, we have the decomposition

$$I_2^{(\gamma)} = - \sum_{i=1}^4 \int \frac{1}{n} \partial_x^\gamma \phi_R^\varepsilon D_i =: \sum_{i=1}^4 I_{2i}^{(\gamma)}. \quad (3.23)$$

Estimate of $I_{21}^{(\gamma)}$. By integration by parts, we obtain

$$I_{21}^{(\gamma)} = - \int \frac{1}{n} \partial_x^\gamma \phi_R^\varepsilon \partial_t \partial_x^\gamma \phi_R^\varepsilon \\ = - \frac{1}{2} \frac{d}{dt} \int \frac{1}{n} |\partial_x^\gamma \phi_R^\varepsilon|^2 + \frac{1}{2} \int \partial_t \left[\frac{1}{n} \right] |\partial_x^\gamma \phi_R^\varepsilon|^2,$$

where the second term on the RHS is bounded by Lemma 3.2

$$\frac{1}{2} \int \partial_t \left[\frac{1}{n} \right] |\partial_x^\gamma \phi_R^\varepsilon|^2 = - \frac{1}{2} \int \left[\frac{\varepsilon \partial_t \tilde{n} + \varepsilon^3 \partial_t n_R^\varepsilon}{n^2} \right] |\partial_x^\gamma \phi_R^\varepsilon|^2 \\ \leq C \varepsilon \|\partial_x^\gamma \phi_R^\varepsilon\|^2 + \varepsilon^3 \|\varepsilon \partial_t n_R^\varepsilon\|^2 \|\partial_x^\gamma \phi_R^\varepsilon\|^2 + \varepsilon \|\partial_x^\gamma \phi_R^\varepsilon\|_{L^\infty}^2 \\ \leq C_1(1 + \varepsilon^2 \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2)(\varepsilon \|\phi_R^\varepsilon\|_{H^{\gamma+1}}^2).$$

Hence

$$I_{21}^{(\gamma)} \leq - \frac{1}{2} \frac{d}{dt} \int \frac{1}{n} |\partial_x^\gamma \phi_R^\varepsilon|^2 + C_1(1 + \varepsilon^2 \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2)(\varepsilon \|\phi_R^\varepsilon\|_{H^{\gamma+1}}^2). \quad (3.24)$$

Estimate of $I_{22}^{(\gamma)}$. By integration by parts, we have

$$I_{22}^{(\gamma)} = \int \frac{\varepsilon}{n} \partial_x^\gamma \phi_R^\varepsilon \partial_t \partial_x^{\gamma+2} \phi_R^\varepsilon \\ = - \int \frac{\varepsilon}{n} \partial_x^{\gamma+1} \phi_R^\varepsilon \partial_t \partial_x^{\gamma+1} \phi_R^\varepsilon - \int \partial_x \left[\frac{\varepsilon}{n} \right] \partial_x^\gamma \phi_R^\varepsilon \partial_t \partial_x^{\gamma+1} \phi_R^\varepsilon \\ =: I_{221}^{(\gamma)} + I_{222}^{(\gamma)}.$$

The first term is estimated by Sobolev embedding, Lemma 3.3 and 3.2 as

$$\begin{aligned}
I_{221}^{(\gamma)} &= -\frac{1}{2} \frac{d}{dt} \int \frac{\varepsilon}{n} |\partial_x^{\gamma+1} \phi_R^\varepsilon|^2 + \frac{1}{2} \int \partial_t \left[\frac{\varepsilon}{n} \right] |\partial_x^{\gamma+1} \phi_R^\varepsilon|^2 \\
&\leq -\frac{1}{2} \frac{d}{dt} \int \frac{\varepsilon}{n} |\partial_x^{\gamma+1} \phi_R^\varepsilon|^2 + C\varepsilon(1 + \varepsilon^2 \|\partial_t n_R^\varepsilon\|_{L^\infty}) (\varepsilon \|\partial_x^{\gamma+1} \phi_R^\varepsilon\|^2) \\
&\leq -\frac{1}{2} \frac{d}{dt} \int \frac{\varepsilon}{n} |\partial_x^{\gamma+1} \phi_R^\varepsilon|^2 + C_1 \varepsilon (1 + \varepsilon \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon) (\varepsilon \|\partial_x^{\gamma+1} \phi_R^\varepsilon\|^2),
\end{aligned} \tag{3.25}$$

where $\|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon$ is defined in (3.2).

For $I_{222}^{(\gamma)}$, integration by parts yields

$$I_{222}^{(\gamma)} = \underbrace{\int \partial_x^{\gamma+1} \phi_R^\varepsilon \partial_x \left[\frac{\varepsilon}{n} \right] \partial_t \partial_x^\gamma \phi_R^\varepsilon}_{:= \mathcal{B}^{(\gamma)}} + \underbrace{\int \partial_x^\gamma \phi_R^\varepsilon \partial_x^2 \left[\frac{\varepsilon}{n} \right] \partial_t \partial_x^\gamma \phi_R^\varepsilon}_{I_{2221}^{(\gamma)}}. \tag{3.26}$$

We first bound $\mathcal{B}^{(\gamma)}$ in (3.26). Since $0 \leq \gamma \leq 2$, by Lemma 3.3 with $\alpha = 1$, and Lemma 3.2 and 3.1, we have

$$\begin{aligned}
\mathcal{B}^{(\gamma)} &= - \int \varepsilon \left(\frac{\partial_x \tilde{n} + \varepsilon^2 \partial_x n_R^\varepsilon}{n^2} \right) \partial_x^{\gamma+1} \phi_R^\varepsilon (\varepsilon \partial_t \partial_x^\gamma \phi_R^\varepsilon) dx \\
&\leq C\varepsilon \|\varepsilon \partial_t \partial_x^\gamma \phi_R^\varepsilon\|^2 + C(1 + \varepsilon^4 \|\partial_x n_R^\varepsilon\|_{L^\infty}^2) (\varepsilon \|\partial_x^{\gamma+1} \phi_R^\varepsilon\|^2) \\
&\leq C_1(1 + \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2) + C_1(1 + \varepsilon^2 \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2) (\varepsilon \|\phi_R^\varepsilon\|_{H^{\gamma+1}}^2).
\end{aligned} \tag{3.27}$$

We now estimate $I_{2221}^{(\gamma)}$ in (3.26). Note that

$$\left| \partial_x^2 \left[\frac{\varepsilon}{n} \right] \right| \leq C\varepsilon^2 (1 + \varepsilon^2 |\partial_x^2 n_R^\varepsilon| + \varepsilon^3 |\partial_x n_R^\varepsilon| + \varepsilon^5 |\partial_x n_R^\varepsilon|^2).$$

By Hölder inequality and Lemma 3.3 with $\alpha = 1$,

$$\begin{aligned}
\varepsilon^2 \int |\partial_x^\gamma \phi_R^\varepsilon| |\partial_t \partial_x^\gamma \phi_R^\varepsilon| &\leq C\varepsilon \|\partial_x^\gamma \phi_R^\varepsilon\|^2 + C\varepsilon \|\varepsilon \partial_t \partial_x^\gamma \phi_R^\varepsilon\|^2 \\
&\leq C\varepsilon \|\partial_x^\gamma \phi_R^\varepsilon\|^2 + C_1(1 + \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2).
\end{aligned} \tag{3.28}$$

By Lemma 3.1, Lemma 3.3 and 3.2,

$$\begin{aligned}
\varepsilon^4 \int |\partial_x^\gamma \phi_R^\varepsilon| |\partial_x^2 n_R^\varepsilon| |\partial_t \partial_x^\gamma \phi_R^\varepsilon| \\
\leq C\varepsilon^2 \|\partial_x^2 n_R^\varepsilon\|^2 (\varepsilon \|\partial_x^\gamma \phi_R^\varepsilon\|_{L^\infty}^2) + C\varepsilon^2 (\varepsilon \|\varepsilon \partial_t \partial_x^\gamma \phi_R^\varepsilon\|^2) \\
\leq C_1 \varepsilon^2 \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2 (\varepsilon \|\phi_R^\varepsilon\|_{H^{\gamma+1}}^2) + C_1 \varepsilon^2 (1 + \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2).
\end{aligned} \tag{3.29}$$

By Hölder inequality and Lemma 3.3, 3.2 and 3.1

$$\begin{aligned}
\varepsilon^5 \int |\partial_x^\gamma \phi_R^\varepsilon| (|\partial_x n_R^\varepsilon| + \varepsilon^2 |\partial_x n_R^\varepsilon|^2) |\partial_t \partial_x^\gamma \phi_R^\varepsilon| \\
\leq \varepsilon^2 (1 + \|\partial_x n_R^\varepsilon\|_{L^\infty}^2) (\varepsilon \|\varepsilon \partial_t \partial_x^\gamma \phi_R^\varepsilon\|^2) + C\varepsilon^2 \|\partial_x n_R^\varepsilon\|_{L^\infty}^2 \|\partial_x^\gamma \phi_R^\varepsilon\|^2 \\
\leq C_1(1 + \varepsilon^2 \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2) (1 + \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2).
\end{aligned} \tag{3.30}$$

Summarizing (3.26), (3.28)-(3.30), we have

$$I_{2221}^{(\gamma)} \leq C_1(1 + \varepsilon^2 \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2) (1 + \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2). \tag{3.31}$$

Therefore, by (3.25), (3.27) and (3.31), we obtain

$$I_{22}^{(\gamma)} \leq -\frac{1}{2} \frac{d}{dt} \int \frac{\varepsilon}{n} |\partial_x^{\gamma+1} \phi_R^\varepsilon|^2 + C_1(1 + \varepsilon^2 \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2)(1 + \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2).$$

Estimate of $I_{23}^{(\gamma)}$. The estimate for $I_{23}^{(\gamma)}$ in (3.23) is no more difficult than that of $I_{21}^{(\gamma)}$ and can be bounded by

$$I_{23}^{(\gamma)} \leq -\frac{1}{2} \frac{d}{dt} \int \frac{\varepsilon \phi^{(1)}}{n} |\partial_x^\gamma \phi_R^\varepsilon|^2 + C(1 + \varepsilon \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2)(\varepsilon \|\phi_R^\varepsilon\|_{H^3}^2).$$

Estimate of $I_{24}^{(\gamma)}$. By using Lemma A.1, and then Lemma 3.3 with $\alpha = 1$ and Lemma 3.2, we have

$$\begin{aligned} I_{24}^{(\gamma)} &= - \int \frac{\varepsilon^2}{n} \partial_x^\gamma \phi_R^\varepsilon \partial_t \partial_x^\gamma \mathcal{R}_3 \\ &\leq C \|\partial_x^\gamma \phi_R^\varepsilon\|^2 + \varepsilon C (\|\phi^{(i)}\|_{H^{\bar{s}_i}}, \varepsilon \|\phi_R^\varepsilon\|_{H^2}) (\varepsilon \|\partial_t \phi_R^\varepsilon\|_{H^\gamma}^2) \\ &\leq C_1(1 + \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2). \end{aligned}$$

The proof of Lemma 3.5 is then complete. \square

When $\gamma = 2$, by extracting the term $\mathcal{B}^{(2)}$ from (3.26), we have the following

Corollary 3.1. *Let $(n_R^\varepsilon, u_R^\varepsilon, \phi_R^\varepsilon)$ be a solution to (3.1) and $0 \leq \gamma \leq 2$, then*

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \left[\|\partial_x^2 u_R^\varepsilon\|^2 \right] + \frac{1}{2} \frac{d}{dt} \left[\left(\int \frac{1 + \varepsilon \phi^{(1)}}{n} |\partial_x^2 \phi_R^\varepsilon|^2 + \int \frac{\varepsilon}{n} |\partial_x^3 \phi_R^\varepsilon|^2 \right) \right] \\ &\leq C_1(1 + \varepsilon^2 \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2)(1 + \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2) + \mathcal{B}^{(2)}, \end{aligned}$$

where

$$\mathcal{B}^{(2)} = \int \partial_x \left[\frac{\varepsilon}{n} \right] \partial_x^3 \phi_R^\varepsilon \partial_t \partial_x^2 \phi_R^\varepsilon.$$

Proof. This follows from (3.14) with $\gamma = 2$. \square

We remark that the precise form of $\mathcal{B}^{(2)}$ is very important for us to close the proof later. Indeed, when $\gamma = 3$, the term $\mathcal{B}^{(3)}$ is not controllable in terms of $\|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon$. We need an exact cancellation by $\mathcal{B}^{(2)} + \varepsilon \mathcal{B}^{(3)}$ (see Remark 3.8 below). This is the reason why the third order derivatives are estimated separately.

3.2. Third order estimates.

Proposition 3.2. *Let $(n_R^\varepsilon, u_R^\varepsilon, \phi_R^\varepsilon)$ be a solution to (3.1), then*

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \left[\varepsilon \|\partial_x^3 u_R^\varepsilon\|^2 \right] + \frac{1}{2} \frac{d}{dt} \left[\left(\int \frac{\varepsilon(1 + \varepsilon \phi^{(1)})}{n} |\partial_x^3 \phi_R^\varepsilon|^2 + \int \frac{\varepsilon^2}{n} |\partial_x^4 \phi_R^\varepsilon|^2 \right) \right] \\ &\leq C_1(1 + \varepsilon^2 \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2)(1 + \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2) + \mathcal{B}^{(3 \times \varepsilon)}, \end{aligned}$$

where

$$\mathcal{B}^{(3 \times \varepsilon)} = - \int \partial_x^3 \phi_R^\varepsilon \partial_x \left[\frac{\varepsilon^2}{n} \right] \partial_t \partial_x^4 \phi_R^\varepsilon.$$

Proof. Taking ∂_x^3 of (3.1b) and then taking inner product with $\varepsilon \partial_x^3 u_R^\varepsilon$, we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\varepsilon \|\partial_x^3 u_R^\varepsilon\|^2) - \int \partial_x^4 u_R^\varepsilon \partial_x^3 u_R^\varepsilon + \int \varepsilon \partial_x^3 [(\tilde{u} + \varepsilon u_R^\varepsilon) \partial_x u_R^\varepsilon] \partial_x^3 u_R^\varepsilon \\ & \quad + \int \varepsilon \partial_x^3 [\partial_x \tilde{u} u_R^\varepsilon] \partial_x^3 u_R^\varepsilon + \int \partial_x^3 [\varepsilon^2 \mathcal{R}_2] \partial_x^3 u_R^\varepsilon \\ & = \int \varepsilon \partial_x^3 \phi_R^\varepsilon \frac{\partial_x^4 u_R^\varepsilon}{\varepsilon} =: I^{(3 \times \varepsilon)}. \end{aligned} \quad (3.32)$$

Estimate of LHS of (3.32). The second term on the LHS of (3.32) vanishes by integration by parts. For the third term, by expanding the derivatives and then integration by parts, we have

$$\begin{aligned} & \int \varepsilon \partial_x^3 [(\tilde{u} + \varepsilon^2 u_R^\varepsilon) \partial_x u_R^\varepsilon] \partial_x^3 u_R^\varepsilon \\ & = \frac{5}{2} \int \varepsilon \partial_x (\tilde{u} + \varepsilon^2 u_R^\varepsilon) |\partial_x^3 u_R^\varepsilon|^2 + \sum_{\beta=2,3} C_3^\beta \int \varepsilon \partial_x^\beta [(\tilde{u} + \varepsilon^2 u_R^\varepsilon)] \partial_x^{4-\beta} u_R^\varepsilon \partial_x^3 u_R^\varepsilon \\ & =: L_{31}^{(3 \times \varepsilon)} + L_{32}^{(3 \times \varepsilon)}. \end{aligned} \quad (3.33)$$

The first term $L_{31}^{(3 \times \varepsilon)}$ on the RHS of (3.33) is estimated as

$$\begin{aligned} L_{31}^{(3 \times \varepsilon)} & \leq C(1 + \varepsilon \|\partial_x u_R^\varepsilon\|_{L^\infty}) (\varepsilon \|\partial_x^3 u_R^\varepsilon\|^2) \\ & \leq C(1 + \varepsilon \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon) (\varepsilon \|\partial_x^3 u_R^\varepsilon\|^2). \end{aligned} \quad (3.34)$$

When $\beta = 3$, the second term $L_{32}^{(3 \times \varepsilon)}$ on the RHS of (3.33) is estimated similarly to (3.34). When $\beta = 2$, the second term $L_{32}^{(3 \times \varepsilon)}$ is estimated

$$\begin{aligned} L_{32}^{(3 \times \varepsilon)} & = \int \varepsilon (\partial_x^2 \tilde{u} + \varepsilon^2 \partial_x^2 u_R^\varepsilon) \partial_x^2 u_R^\varepsilon \partial_x^3 u_R^\varepsilon \\ & \leq C(1 + \varepsilon^2 \|\partial_x^2 u_R^\varepsilon\|_{L^\infty}) (\varepsilon \|\partial_x^2 u_R^\varepsilon\| \|\partial_x^3 u_R^\varepsilon\|) \\ & \leq C(1 + \varepsilon \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon) (\varepsilon \|u_R^\varepsilon\|_{H^3}^2). \end{aligned}$$

By Lemma A.1, the last two terms on the LHS of (3.32) are easily bounded by

$$\varepsilon (1 + \|u_R^\varepsilon\|_{H^3}^2).$$

Hence, the last four terms on the LHS of (3.32) are bounded by

$$C(1 + \varepsilon \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon) (1 + \varepsilon \|u_R^\varepsilon\|_{H^3}^2).$$

Decomposition of $I^{(3 \times \varepsilon)}$ in (3.32). Taking ∂_x^3 of (3.1a), we have

$$\begin{aligned} \frac{\partial_x^4 u_R^\varepsilon}{\varepsilon} = & \frac{1}{n} \left[\frac{(1-u)}{\varepsilon} \partial_x^4 n_R^\varepsilon - \partial_t \partial_x^3 n_R^\varepsilon - \sum_{\beta=1}^3 C_3^\beta \partial_x^\beta (\tilde{n} + \varepsilon^2 n_R^\varepsilon) \partial_x^{4-\beta} u_R^\varepsilon \right. \\ & - \sum_{\beta=1}^3 C_3^\beta \partial_x^\beta (\tilde{u} + \varepsilon^2 u_R^\varepsilon) \partial_x^{4-\beta} n_R^\varepsilon - \sum_{\beta=0}^3 C_3^\beta \partial_x^\beta u_R^\varepsilon \partial_x^{4-\beta} \tilde{n} \\ & \left. - \sum_{\beta=0}^3 C_3^\beta \partial_x^\beta n_R^\varepsilon \partial_x^{4-\beta} \tilde{u} - \varepsilon \partial_x^3 \mathcal{R}_1 \right] = \sum_{i=1}^7 A_i^{(3)}. \end{aligned} \quad (3.35)$$

Accordingly $I^{(3 \times \varepsilon)}$ is decomposed into

$$I^{(3 \times \varepsilon)} = \sum_{i=1}^7 \int \varepsilon \partial_x^3 \phi_R^\varepsilon A_i^{(3)} = \sum_{i=1}^7 I_i^{(3 \times \varepsilon)}. \quad (3.36)$$

Estimate of $I_i^{(3 \times \varepsilon)}$ for $3 \leq i \leq 7$. By a direct computation, $I_3^{(3 \times \varepsilon)}$ takes the form

$$I_3^{(3 \times \varepsilon)} = - \sum_{\beta=1}^3 C_3^\beta \int \frac{\varepsilon}{n} \partial_x^3 \phi_R^\varepsilon \partial_x^\beta \tilde{n} \partial_x^{4-\beta} u_R^\varepsilon - \sum_{\beta=1}^3 C_3^\beta \int \frac{\varepsilon^3}{n} \partial_x^3 \phi_R^\varepsilon \partial_x^\beta n_R^\varepsilon \partial_x^{4-\beta} u_R^\varepsilon. \quad (3.37)$$

The first term on the RHS is bilinear in $(n_R^\varepsilon, u_R^\varepsilon)$ and is bounded by

$$C\varepsilon \|\partial_x^3 \phi_R^\varepsilon\|^2 + C(\|u_R^\varepsilon\|_{H^2}^2 + \varepsilon \|\partial_x^3 u_R^\varepsilon\|^2).$$

For the second term on the RHS of (3.37), when $\beta = 1, 2$, it is bounded by Lemma 3.1

$$\begin{aligned} \int \frac{\varepsilon^3}{n} \partial_x^3 \phi_R^\varepsilon \partial_x^\beta n_R^\varepsilon \partial_x^{4-\beta} u_R^\varepsilon & \leq C\varepsilon^2 \|\partial_x^\beta n_R^\varepsilon\|^2 (\varepsilon^2 \|\partial_x^3 \phi_R^\varepsilon\|_{L^\infty}^2) + C\varepsilon^2 \|\partial_x^{4-\beta} u_R^\varepsilon\|^2 \\ & \leq C_1(1 + \varepsilon^2 \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2)(\varepsilon \|u_R^\varepsilon\|_{H^3}^2 + \varepsilon^2 \|\phi_R^\varepsilon\|_{H^4}^2). \end{aligned}$$

When $\beta = 3$, by integration by parts, $H^1 \hookrightarrow L^\infty$ and Lemma 3.2,

$$\begin{aligned} - \int \frac{\varepsilon^3}{n} \partial_x^3 \phi_R^\varepsilon \partial_x^3 n_R^\varepsilon \partial_x u_R^\varepsilon & = \int \frac{\varepsilon^3}{n} \partial_x^4 \phi_R^\varepsilon \partial_x^2 n_R^\varepsilon \partial_x u_R^\varepsilon \\ & + \int \frac{\varepsilon^3}{n} \partial_x^3 \phi_R^\varepsilon \partial_x^2 n_R^\varepsilon \partial_x^2 u_R^\varepsilon + \int \partial_x \left[\frac{\varepsilon^3}{n} \right] \partial_x^3 \phi_R^\varepsilon \partial_x^2 n_R^\varepsilon \partial_x u_R^\varepsilon \\ & \leq C(\varepsilon \|\partial_x^3 \phi_R^\varepsilon\|_{H^1})(\varepsilon \|\partial_x^2 n_R^\varepsilon\|)(\varepsilon \|\partial_x u_R^\varepsilon\|_{H^{1,\infty}})(1 + \varepsilon^3 \|\partial_x n_R^\varepsilon\|_{L^\infty}) \\ & \leq C_1(1 + \varepsilon^2 \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2)(\varepsilon \|\partial_x u_R^\varepsilon\|_{H^2}^2 + \varepsilon^2 \|\phi_R^\varepsilon\|_{H^4}^2). \end{aligned}$$

This completes the estimate of $I_3^{(3 \times \varepsilon)}$. The terms $I_i^{(3 \times \varepsilon)}$ for $i = 4, 5, 6, 7$ can be bounded similarly with the same bound.

In summary, we have

$$\sum_{i=3}^7 I_i^{(3 \times \varepsilon)} \leq C_1(1 + \varepsilon^2 \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2)(1 + \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2).$$

Proposition 3.2 then follows from the following Lemma 3.6, Lemma 3.7 and Proposition 3.3. \square

Lemma 3.6 (*Estimate for $I_1^{(3 \times \varepsilon)}$*). *Let $(n_R^\varepsilon, u_R^\varepsilon, \phi_R^\varepsilon)$ be a solution to (3.1), then*

$$I_1^{(3 \times \varepsilon)} \leq C_1(1 + \varepsilon^2 \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2)(1 + \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2).$$

Proof. Recall from (3.36),

$$I_1^{(3 \times \varepsilon)} = \int \partial_x^3 \phi_R^\varepsilon \left[\frac{(1-u)}{n} \partial_x^4 n_R^\varepsilon \right].$$

Taking ∂_x^4 of (3.1c), we have

$$\partial_x^4 n_R^\varepsilon = \partial_x^4 \left\{ \phi_R^\varepsilon - \varepsilon \partial_x^2 \phi_R^\varepsilon + \varepsilon (\phi^{(1)} \phi_R^\varepsilon) + \varepsilon^2 \mathcal{R}_3 \right\}.$$

Accordingly, we split $I_1^{(3 \times \varepsilon)}$ as

$$I_1^{(3 \times \varepsilon)} = \int \partial_x^3 \phi_R^\varepsilon \left[\frac{(1-u)}{n} \partial_x^4 n_R^\varepsilon \right] = \sum_{i=1}^4 I_{1i}^{(3 \times \varepsilon)}. \quad (3.38)$$

Estimate of $I_{11}^{(3 \times \varepsilon)}$ in (3.38). By integration by parts, we have

$$\begin{aligned} I_{11}^{(3 \times \varepsilon)} &= \int \partial_x^3 \phi_R^\varepsilon \left[\frac{(1-u)}{n} \partial_x^4 \phi_R^\varepsilon \right] \\ &= - \int \partial_x \left[\frac{(1-u)}{n} \right] \|\partial_x^3 \phi_R^\varepsilon\|^2 \\ &\leq C\varepsilon \|\partial_x^3 \phi_R^\varepsilon\|^2 + C\varepsilon^2 (\|\partial_x n_R^\varepsilon\|_{L^\infty} + \|\partial_x u_R^\varepsilon\|_{L^\infty}) (\varepsilon \|\partial_x^3 \phi_R^\varepsilon\|^2) \\ &\leq C_1(1 + \varepsilon^2 \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2) (\varepsilon \|\partial_x^3 \phi_R^\varepsilon\|^2). \end{aligned} \quad (3.39)$$

Estimate of $I_{12}^{(3 \times \varepsilon)}$ in (3.38). By integration by parts twice, we have

$$\begin{aligned} I_{12}^{(3 \times \varepsilon)} &= - \int \varepsilon \partial_x^3 \phi_R^\varepsilon \left[\frac{(1-u)}{n} \partial_x^6 \phi_R^\varepsilon \right] \\ &= - \frac{3}{2} \int \varepsilon \partial_x \left[\frac{(1-u)}{n} \right] |\partial_x^4 \phi_R^\varepsilon|^2 - \int \varepsilon \partial_x^3 \phi_R^\varepsilon \partial_x^2 \left[\frac{(1-u)}{n} \right] \partial_x^4 \phi_R^\varepsilon \\ &=: I_{121}^{(3 \times \varepsilon)} + I_{122}^{(3 \times \varepsilon)}. \end{aligned} \quad (3.40)$$

For the first term $I_{121}^{(3 \times \varepsilon)}$, since

$$\partial_x \left(\frac{(1-u)}{n} \right) \leq C\varepsilon + C\varepsilon^3 (|\partial_x u_R^\varepsilon| + \|\partial_x n_R^\varepsilon\|_{L^\infty}),$$

by Hölder inequality, Sobolev embedding and Lemma 3.2, we deduce

$$\begin{aligned} I_{121}^{(3 \times \varepsilon)} &\leq C\varepsilon^2 \|\partial_x^4 \phi_R^\varepsilon\|^2 + C\varepsilon^2 (\|\partial_x u_R^\varepsilon\|_{L^\infty} + \|\partial_x n_R^\varepsilon\|_{L^\infty}) (\varepsilon^2 \|\partial_x^4 \phi_R^\varepsilon\|^2) \\ &\leq C(1 + \varepsilon^2 \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2) (\varepsilon^2 \|\partial_x^4 \phi_R^\varepsilon\|^2). \end{aligned} \quad (3.41)$$

We note that

$$\begin{aligned} \left| \partial_x^2 \left[\frac{(1-u)}{n} \right] \right| &\leq C \left(\varepsilon + \varepsilon^3 (|\partial_x^2 n_R^\varepsilon| + |\partial_x^2 u_R^\varepsilon|) \right. \\ &\quad \left. + \varepsilon^4 (|\partial_x n_R^\varepsilon| + |\partial_x u_R^\varepsilon|) + \varepsilon^6 (|\partial_x n_R^\varepsilon|^2 + |\partial_x u_R^\varepsilon|^2) \right). \end{aligned} \quad (3.42)$$

To estimate $I_{122}^{(3 \times \varepsilon)}$ in (3.40), we first observe

$$\int \varepsilon^2 |\partial_x^3 \phi_R^\varepsilon| |\partial_x^4 \phi_R^\varepsilon| \leq C (\varepsilon \|\partial_x^3 \phi_R^\varepsilon\|^2 + \varepsilon^2 \|\partial_x^4 \phi_R^\varepsilon\|^2). \quad (3.43)$$

Secondly, by Sobolev embedding, and Lemma 3.1

$$\begin{aligned} &\int \varepsilon^4 \partial_x^3 \phi_R^\varepsilon (|\partial_x^2 u_R^\varepsilon| + |\partial_x^2 n_R^\varepsilon|) \partial_x^4 \phi_R^\varepsilon \\ &\leq C \varepsilon^2 (\|\partial_x^2 u_R^\varepsilon\| + \|\partial_x^2 n_R^\varepsilon\|) (\varepsilon \|\partial_x^3 \phi_R^\varepsilon\|_{L^\infty}) (\varepsilon \|\partial_x^4 \phi_R^\varepsilon\|) \\ &\leq C_1 (1 + \varepsilon \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon) (\varepsilon^2 \|\phi_R^\varepsilon\|_{H^4}^2). \end{aligned} \quad (3.44)$$

Furthermore, by Sobolev embedding, and Lemma 3.1

$$\begin{aligned} \|\partial_x n_R^\varepsilon\|_{L^\infty}^2 + \|\partial_x u_R^\varepsilon\|_{L^\infty}^2 &\leq C (\|\partial_x n_R^\varepsilon\|_{H^1}^2 + \|\partial_x u_R^\varepsilon\|_{H^1}^2) \\ &\leq C_1 (1 + \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2), \end{aligned}$$

it is easy to bound

$$\begin{aligned} &\int \varepsilon^5 \partial_x^3 \phi_R^\varepsilon (|\partial_x n_R^\varepsilon| + |\partial_x u_R^\varepsilon|) + \varepsilon^2 (|\partial_x n_R^\varepsilon|^2 + |\partial_x u_R^\varepsilon|^2) \partial_x^4 \phi_R^\varepsilon \\ &\leq C_1 (1 + \varepsilon^2 \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2) (\varepsilon^2 \|\phi_R^\varepsilon\|_{H^4}^2). \end{aligned} \quad (3.45)$$

By (3.42)-(3.45), the term $I_{122}^{(3 \times \varepsilon)}$ in (3.40) is bounded by

$$I_{12}^{(3 \times \varepsilon)} \leq C_1 (1 + \varepsilon^2 \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2) (1 + \varepsilon \|\phi_R^\varepsilon\|_{H^3}^2 + \varepsilon^2 \|\phi_R^\varepsilon\|_{H^4}^2). \quad (3.46)$$

By (3.41) and (3.46), we have

$$I_{12}^{(3 \times \varepsilon)} \leq C_1 (1 + \varepsilon^2 \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2) (1 + \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2).$$

Estimate of $I_{13}^{(3 \times \varepsilon)}$ in (3.38). It is bounded similarly to $I_{11}^{(3 \times \varepsilon)}$ in (3.40),

$$I_{13}^{(3 \times \varepsilon)} \leq C_1 (1 + \varepsilon^2 \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2) (\|\phi_R^\varepsilon\|_{H^2}^2 + \varepsilon \|\partial_x^3 \phi_R^\varepsilon\|^2).$$

Estimate of $I_{14}^{(3 \times \varepsilon)}$ in (3.38). Recall that $|1-u|/n$ is uniformly bounded and from (1.16),

$$\left| \partial_x \left[\frac{(1-u)}{n} \right] \right| \leq C \varepsilon (1 + \varepsilon^2 (\|\partial_x n_R^\varepsilon\|_{L^\infty} + \|\partial_x u_R^\varepsilon\|_{L^\infty})).$$

By using Lemma A.1, we then have

$$\begin{aligned}
I_{14}^{(3 \times \varepsilon)} &= \int \partial_x^3 \phi_R^\varepsilon \left[\frac{\varepsilon^2(1-u)}{n} \partial_x^4 \mathcal{R}_3 \right] \\
&= - \int \partial_x^4 \phi_R^\varepsilon \left[\frac{\varepsilon^2(1-u)}{n} \right] \partial_x^3 \mathcal{R}_3 - \int \partial_x^3 \phi_R^\varepsilon \partial_x \left[\frac{\varepsilon^2(1-u)}{n} \right] \partial_x^3 \mathcal{R}_3 \\
&\leq C(\|\phi^{(i)}\|_{H^{\tilde{s}_i}}, \varepsilon \|\phi_R^\varepsilon\|_{H^3})(\varepsilon^2 \|\phi_R^\varepsilon\|_{H^3}^2) \\
&\quad + C(1 + \varepsilon^2 \|(\partial_x n_R^\varepsilon, \partial_x u_R^\varepsilon)\|_{L^\infty}^2)(\varepsilon \|\partial_x^3 \phi_R^\varepsilon\|^2 + \varepsilon^2 \|\partial_x^4 \phi_R^\varepsilon\|^2) \\
&\leq C_1(1 + \varepsilon^2 \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2)(1 + \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2).
\end{aligned}$$

By combining the estimates for $I_{1i}^{(3 \times \varepsilon)}$ ($1 \leq i \leq 4$) together, we complete the proof of Lemma 3.6. \square

Lemma 3.7 (Estimate for $I_2^{(3 \times \varepsilon)}$). *Let $(n_R^\varepsilon, u_R^\varepsilon, \phi_R^\varepsilon)$ be a solution to (3.1), then*

$$\begin{aligned}
I_2^{(3 \times \varepsilon)} &\leq - \frac{1}{2} \frac{d}{dt} \left[\left(\int \frac{\varepsilon(1 + \varepsilon \phi^{(1)})}{n} |\partial_x^3 \phi_R^\varepsilon|^2 + \int \frac{\varepsilon^2}{n} |\partial_x^4 \phi_R^\varepsilon|^2 \right) \right] \\
&\leq C_1(1 + \varepsilon^2 \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2)(1 + \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2) + \mathcal{B}^{(3 \times \varepsilon)},
\end{aligned}$$

where

$$\mathcal{B}^{(3 \times \varepsilon)} = - \int \partial_x^3 \phi_R^\varepsilon \partial_x \left[\frac{\varepsilon^2}{n} \right] \partial_t \partial_x^4 \phi_R^\varepsilon.$$

Proof. We first recall that from (3.36)

$$I_2^{(3 \times \varepsilon)} = - \int \frac{\varepsilon}{n} \partial_x^3 \phi_R^\varepsilon \partial_t \partial_x^3 n_R^\varepsilon.$$

Taking $\partial_t \partial_x^3$ of (3.1c), and then inserting the result in $I_2^{(3 \times \varepsilon)}$, we have

$$I_2^{(3 \times \varepsilon)} = - \int \frac{\varepsilon}{n} \partial_x^3 \phi_R^\varepsilon \partial_t \partial_x^3 \left[\phi_R^\varepsilon - \varepsilon \partial_x^2 \phi_R^\varepsilon + \varepsilon(\phi^{(1)} \phi_R^\varepsilon) + \varepsilon^2 \mathcal{R}_3 \right] =: \sum_{i=1}^4 I_{2i}^{(3 \times \varepsilon)}. \quad (3.47)$$

Estimate of $I_{21}^{(3 \times \varepsilon)}$ in (3.47). By integration by parts in t , and then using Sobolev embedding $H^1 \hookrightarrow L^\infty$ and Lemma 3.2, we have

$$\begin{aligned}
I_{21}^{(3 \times \varepsilon)} &= - \int \frac{\varepsilon}{n} \partial_x^3 \phi_R^\varepsilon \partial_t \partial_x^3 \phi_R^\varepsilon \\
&= - \frac{1}{2} \frac{d}{dt} \int \frac{\varepsilon}{n} |\partial_x^3 \phi_R^\varepsilon|^2 + \frac{1}{2} \int \partial_t \left[\frac{\varepsilon}{n} \right] |\partial_x^3 \phi_R^\varepsilon|^2 \\
&= - \frac{1}{2} \frac{d}{dt} \int \frac{\varepsilon}{n} |\partial_x^3 \phi_R^\varepsilon|^2 - \int \varepsilon \left[\frac{\varepsilon \partial_t \tilde{n}}{n^2} + \frac{\varepsilon^3 \partial_t n_R^\varepsilon}{n^2} \right] |\partial_x^3 \phi_R^\varepsilon|^2 \\
&\leq - \frac{1}{2} \frac{d}{dt} \int \frac{\varepsilon}{n} |\partial_x^3 \phi_R^\varepsilon|^2 + C\varepsilon(1 + \|\varepsilon \partial_t n_R^\varepsilon\|_{L^\infty})(\varepsilon \|\partial_x^3 \phi_R^\varepsilon\|^2) \\
&\leq - \frac{1}{2} \frac{d}{dt} \int \frac{\varepsilon}{n} |\partial_x^3 \phi_R^\varepsilon|^2 + C_1 \varepsilon(1 + \varepsilon \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon)(\varepsilon \|\partial_x^3 \phi_R^\varepsilon\|^2).
\end{aligned} \quad (3.48)$$

Estimate of $I_{22}^{(3 \times \varepsilon)}$ in (3.47). By integration by parts, we have

$$\begin{aligned} I_{22}^{(3 \times \varepsilon)} &= \int \partial_x^3 \phi_R^\varepsilon \left[\frac{\varepsilon^2}{n} \right] \partial_t \partial_x^5 \phi_R^\varepsilon \\ &= - \underbrace{\int \partial_x^4 \phi_R^\varepsilon \left[\frac{\varepsilon^2}{n} \right] \partial_t \partial_x^4 \phi_R^\varepsilon}_{I_{221}^{(3 \times \varepsilon)}} - \underbrace{\int \partial_x^3 \phi_R^\varepsilon \partial_x \left[\frac{\varepsilon^2}{n} \right] \partial_t \partial_x^4 \phi_R^\varepsilon}_{\mathcal{B}^{(3 \times \varepsilon)}}. \end{aligned} \quad (3.49)$$

For the first term $I_{221}^{(3 \times \varepsilon)}$ in (3.49), we have

$$\begin{aligned} I_{221}^{(3 \times \varepsilon)} &= -\frac{1}{2} \frac{d}{dt} \int \left[\frac{\varepsilon^2}{n} \right] |\partial_x^4 \phi_R^\varepsilon|^2 + \frac{1}{2} \int \partial_t \left[\frac{\varepsilon^2}{n} \right] |\partial_x^4 \phi_R^\varepsilon|^2 \\ &= -\frac{1}{2} \frac{d}{dt} \int \left[\frac{\varepsilon^2}{n} \right] |\partial_x^4 \phi_R^\varepsilon|^2 - \frac{1}{2} \int \varepsilon^2 \left[\frac{\varepsilon \partial_t \tilde{n}}{n^2} + \frac{\varepsilon^3 \partial_t n_R^\varepsilon}{n^2} \right] |\partial_x^4 \phi_R^\varepsilon|^2 \\ &\leq -\frac{1}{2} \frac{d}{dt} \int \left[\frac{\varepsilon^2}{n} \right] |\partial_x^4 \phi_R^\varepsilon|^2 + C_1 \varepsilon (1 + \varepsilon \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon) (\varepsilon^2 \|\partial_x^4 \phi_R^\varepsilon\|^2). \end{aligned}$$

The term $\mathcal{B}^{(3 \times \varepsilon)}$ cannot be controlled in terms of $\|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon$ so far (see Remark 3.8). Its estimate is postponed to Section 3.3 by an exact cancellation with $\mathcal{B}^{(2)}$ in Corollary 3.1.

Estimate of $I_{23}^{(3 \times \varepsilon)}$ in (3.47). Similar to the estimate of $I_{21}^{(3 \times \varepsilon)}$ in (3.48), we have

$$I_{23}^{(3 \times \varepsilon)} \leq -\frac{1}{2} \frac{d}{dt} \int \frac{\varepsilon^2 \phi^{(1)}}{n} |\partial_x^3 \phi_R^\varepsilon|^2 + C_1 \varepsilon (1 + \varepsilon \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon) (1 + \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon).$$

Estimate of $I_{24}^{(3 \times \varepsilon)}$ in (3.47). Integration by parts yields

$$\begin{aligned} I_{24}^{(3 \times \varepsilon)} &= - \int \partial_x^3 \phi_R^\varepsilon \left[\frac{\varepsilon^3}{n} \right] \partial_t \partial_x^3 \mathcal{R}_3 \\ &= \int \partial_x^4 \phi_R^\varepsilon \left[\frac{\varepsilon^3}{n} \right] \partial_t \partial_x^2 \mathcal{R}_3 + \int \partial_x^3 \phi_R^\varepsilon \partial_x \left[\frac{\varepsilon^3}{n} \right] \partial_t \partial_x^2 \mathcal{R}_3. \end{aligned} \quad (3.50)$$

By Lemma A.1 in the Appendix,

$$\varepsilon \|\varepsilon \partial_t \partial_x^2 \mathcal{R}_3\|^2 \leq \varepsilon C (\|\phi^{(i)}\|_{H^{\tilde{s}_i}}, \varepsilon \|\phi_R^\varepsilon\|_{H^2}) \|\varepsilon \partial_t \phi_R^\varepsilon\|_{H^2}$$

and by Sobolev embedding and Lemma 3.1

$$\|\partial_x (\frac{1}{n})\|_{L^\infty}^2 \leq C_1 \varepsilon (1 + \varepsilon^2 \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2).$$

By (3.50), we therefore have

$$\begin{aligned} I_{24}^{(3 \times \varepsilon)} &\leq C (1 + \varepsilon^2 \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2) (\varepsilon^2 \|\partial_x^3 \phi_R^\varepsilon\|_{H^1}^2 + C_1 \varepsilon (\varepsilon \|\varepsilon \partial_t \phi_R^\varepsilon\|_{H^2}^2) \\ &\leq C_1 (1 + \varepsilon^2 \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2) (1 + \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2). \end{aligned} \quad (3.51)$$

Lemma 3.7 then follows. \square

Remark 3.8. By Lemma 3.3, only $\|\partial_t \partial_x^2 \phi_R^\varepsilon\|_{L^2}$ can be controlled in terms of $\|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2$ through $\|\partial_t \phi_R^\varepsilon\|_{H^1}^2$ by Lemma 3.2. However, upon integration by parts, there will be a contribution $\int \partial_x^5 \phi_R^\varepsilon \partial_x \left[\frac{\varepsilon^2}{n} \right] \partial_t \partial_x^2 \phi_R^\varepsilon$, which is not controllable in terms of $\|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2$ due to $\partial_x^5 \phi_R^\varepsilon$.

However, $\mathcal{B}^{(3 \times \varepsilon)}$ is controlled by an exact cancellation by using (3.1c) one more time. Besides the term $\mathcal{B}^{(3 \times \varepsilon)}$, there is a term $\mathcal{B}^{(2)}$ with the same structure in Corollary 3.1. Recalling $\mathcal{B}^{(2)}$ in Corollary 3.1, we obtain

$$\mathcal{G}^{(2, \varepsilon)} = \mathcal{B}^{(2)} + \mathcal{B}^{(3 \times \varepsilon)} = \int \partial_x \left(\frac{\varepsilon}{n} \right) \partial_x^3 \phi_R^\varepsilon \left[\partial_t \partial_x^2 (\phi_R^\varepsilon - \varepsilon \partial_x^2 \phi_R^\varepsilon) \right]. \quad (3.52)$$

The crucial observation is that the combination $(\phi_R^\varepsilon - \varepsilon \partial_x^2 \phi_R^\varepsilon)$ exactly appears in the Poisson equation (3.1c) and can be controlled.

3.3. Control of $\mathcal{G}^{(2, \varepsilon)}$.

Proposition 3.3. Let $(n_R^\varepsilon, u_R^\varepsilon, \phi_R^\varepsilon)$ be a solution to (3.1), then

$$\mathcal{G}^{(2, \varepsilon)} \leq C_1 (1 + \varepsilon^2 \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2) (1 + \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2), \quad (3.53)$$

where $\|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon$ is defined in (3.2).

Proof. Recall (3.52). From the Poisson equation (3.1c), we have

$$\begin{aligned} \mathcal{G}^{(2, \varepsilon)} &= \int \partial_x \left(\frac{\varepsilon}{n} \right) \partial_x^3 \phi_R^\varepsilon \left[\partial_t \partial_x^2 (\phi_R^\varepsilon - \varepsilon \partial_x^2 \phi_R^\varepsilon) \right] \\ &= \int \partial_x \left(\frac{\varepsilon}{n} \right) \partial_x^3 \phi_R^\varepsilon \left[\partial_t \partial_x^2 (n_R^\varepsilon - \varepsilon (\phi^{(1)} \phi_R^\varepsilon) - \varepsilon^2 \mathcal{R}_3) \right] = \sum_{i=1}^3 \mathcal{G}_i^{(2, \varepsilon)}. \end{aligned} \quad (3.54)$$

Estimate of $\mathcal{G}_1^{(2, \varepsilon)}$. By integration by parts, we have

$$\begin{aligned} \mathcal{G}_1^{(2, \varepsilon)} &= \int \partial_x^3 \phi_R^\varepsilon \partial_x \left[\frac{\varepsilon}{n} \right] \partial_t \partial_x^2 n_R^\varepsilon \\ &= - \int \partial_x^4 \phi_R^\varepsilon \partial_x \left[\frac{\varepsilon}{n} \right] \partial_{tx} n_R^\varepsilon - \int \partial_x^3 \phi_R^\varepsilon \partial_x^2 \left[\frac{\varepsilon}{n} \right] \partial_{tx} n_R^\varepsilon \\ &=: \mathcal{G}_{11}^{(2, \varepsilon)} + \mathcal{G}_{12}^{(2, \varepsilon)}. \end{aligned} \quad (3.55)$$

Recalling the expression of n in (1.16), we have

$$\left| \partial_x \left(\frac{\varepsilon}{n} \right) \right| \leq C(\varepsilon^2 + \varepsilon^4 |\partial_x n_R^\varepsilon|),$$

and

$$\left| \partial_x^2 \left(\frac{\varepsilon}{n} \right) \right| \leq C(\varepsilon^2 + \varepsilon^4 |\partial_x^2 n_R^\varepsilon| + \varepsilon^5 |\partial_x n_R^\varepsilon| + \varepsilon^7 |\partial_x n_R^\varepsilon|^2).$$

The first term $\mathcal{G}_{11}^{(2, \varepsilon)}$ in (3.55) is bounded by Sobolev embedding, Lemma 3.2 and 3.1

$$\begin{aligned} \mathcal{G}_{11}^{(2, \varepsilon)} &\leq C(\varepsilon \|\partial_x^4 \phi_R^\varepsilon\|) \|\varepsilon \partial_{tx} n_R^\varepsilon\| + C\varepsilon(\varepsilon \|\partial_x^4 \phi_R^\varepsilon\|)(\varepsilon \|\partial_x n_R^\varepsilon\|_{L^\infty}) \|\varepsilon \partial_{tx} n_R^\varepsilon\| \\ &\leq C\varepsilon^2 \|\partial_x^4 \phi_R^\varepsilon\|^2 + C_1 (1 + \varepsilon^2 \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2) (1 + \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2). \end{aligned} \quad (3.56)$$

To estimate $\mathcal{G}_{12}^{(2,\varepsilon)}$ in (3.54), we first observe that by Lemma 3.2 and Lemma 3.1,

$$\begin{aligned} \varepsilon^2 \int |\partial_x^3 \phi_R^\varepsilon \partial_{tx} n_R^\varepsilon| &\leq C\varepsilon \|\partial_x^3 \phi_R^\varepsilon\|^2 + C\|\varepsilon \partial_{tx} n_R^\varepsilon\|^2 \\ &\leq C_1(1 + \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2). \end{aligned} \quad (3.57)$$

Secondly, by Sobolev embedding $H^1 \hookrightarrow L^\infty$, Lemma 3.1 and Lemma 3.2

$$\begin{aligned} \varepsilon^4 \int |\partial_x^3 \phi_R^\varepsilon \partial_x^2 n_R^\varepsilon \partial_{tx} n_R^\varepsilon| &\leq C\varepsilon^2 \|\varepsilon \partial_x^3 \phi_R^\varepsilon\|_{L^\infty}^2 \|\partial_x^2 n_R^\varepsilon\|^2 + C\varepsilon^2 \|\varepsilon \partial_{tx} n_R^\varepsilon\|^2 \\ &\leq C_1(1 + \varepsilon^2 \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2)(1 + \varepsilon^2 \|\phi_R^\varepsilon\|_{H^4}^2). \end{aligned} \quad (3.58)$$

Finally,

$$\begin{aligned} \varepsilon^5 \int |\partial_x^3 \phi_R^\varepsilon| (1 + |\partial_x n_R^\varepsilon|) |\partial_x n_R^\varepsilon| |\partial_{tx} n_R^\varepsilon| \\ \leq C\varepsilon^2 (1 + \|\partial_x n_R^\varepsilon\|_{L^\infty}^2) \|\varepsilon \partial_x^3 \phi_R^\varepsilon\|^2 + C\varepsilon^2 \|\partial_x n_R^\varepsilon\|_{L^\infty}^2 \|\varepsilon \partial_{tx} n_R^\varepsilon\|^2 \\ \leq C_1(1 + \varepsilon^2 \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2)(1 + \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2). \end{aligned} \quad (3.59)$$

Summarizing inequalities (3.57), (3.58) and (3.59), we have

$$\mathcal{G}_{12}^{2,\varepsilon} \leq C_1(1 + \varepsilon^2 \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2)(1 + \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2). \quad (3.60)$$

Combining (3.56) and (3.60), we can bound $\mathcal{G}_1^{2,\varepsilon}$ in (3.55) as

$$\mathcal{G}_1^{2,\varepsilon} \leq C_1(1 + \varepsilon^2 \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2)(1 + \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2). \quad (3.61)$$

Estimate of $\mathcal{G}_2^{(2,\varepsilon)}$ in (3.54). From Lemma 3.3 and 3.2

$$\begin{aligned} \varepsilon \|\varepsilon \partial_t \partial_x^2 (\phi^{(1)} \phi_R^\varepsilon)\|_{L^2}^2 &\leq C_1 \|\varepsilon \partial_t n_R^\varepsilon\|_{H^1}^2 + C_1 \varepsilon \|\phi_R^\varepsilon\|_{H^2}^2 \\ &\leq C_1(1 + \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2), \end{aligned}$$

and

$$\left| \partial_x \left(\frac{\varepsilon^2}{n} \right) \right| \leq C\varepsilon^3(1 + \varepsilon^2 |\partial_x n_R^\varepsilon|).$$

By Hölder inequality and Sobolev embedding, we have

$$\begin{aligned} \mathcal{G}_2^{(2,\varepsilon)} &= \int \partial_x \left(\frac{\varepsilon^2}{n} \right) \partial_x^3 \phi_R^\varepsilon \partial_t \partial_x^2 (\phi^{(1)} \phi_R^\varepsilon) \\ &\leq C\varepsilon(1 + \varepsilon^2 \|\partial_x n\|_{L^\infty}^2) (\varepsilon \|\partial_x^3 \phi_R^\varepsilon\|^2) + C\varepsilon (\varepsilon \|\varepsilon \partial_t \partial_x^2 \phi_R^\varepsilon\|^2) \\ &\leq C_1 \varepsilon (1 + \varepsilon^2 \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2) (1 + \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2). \end{aligned} \quad (3.62)$$

Estimate of $\mathcal{G}_3^{(2,\varepsilon)}$ in (3.54). As the estimate for $I_{24}^{(3 \times \varepsilon)}$ in (3.51), we have

$$\mathcal{G}_3^{(2,\varepsilon)} \leq C_1(1 + \varepsilon^2 \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2)(1 + \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2). \quad (3.63)$$

We complete the proof of Proposition 3.3 by adding the estimates (3.61), (3.62) and (3.63) together. \square

Proof of Theorem 1.3 for $T_i = 0$. Adding the Propositions 3.1 with $\gamma = 0, 1$, Corollary 3.1 and Proposition 3.2 and 3.3 together, we obtain

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} [\|u_R^\varepsilon\|_{H^2}^2 + \varepsilon \|\partial_x^3 u_R^\varepsilon\|_{L^2}^2] + \frac{1}{2} \frac{d}{dt} \left[\left(\int \frac{1 + \varepsilon \phi^{(1)}}{n} |\phi_R^\varepsilon|^2 + \int \frac{\varepsilon}{n} |\partial_x \phi_R^\varepsilon|^2 \right) \right. \\
& \quad + \left(\int \frac{1 + \varepsilon \phi^{(1)}}{n} |\partial_x \phi_R^\varepsilon|^2 + \int \frac{\varepsilon}{n} |\partial_x^2 \phi_R^\varepsilon|^2 \right) + \left(\int \frac{1 + \varepsilon \phi^{(1)}}{n} |\partial_x^2 \phi_R^\varepsilon|^2 \right. \\
& \quad \left. \left. + \int \frac{\varepsilon}{n} |\partial_x^3 \phi_R^\varepsilon|^2 \right) + \left(\int \frac{\varepsilon(1 + \varepsilon \phi^{(1)})}{n} |\partial_x^3 \phi_R^\varepsilon|^2 + \int \frac{\varepsilon^2}{n} |\partial_x^4 \phi_R^\varepsilon|^2 \right) \right] \\
& \leq C_1 (1 + \varepsilon^2 \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2) (1 + \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2).
\end{aligned} \tag{3.64}$$

Since $\phi^{(1)}$ is uniformly bounded, there exists some $\varepsilon_1 > 0$ such that when $\varepsilon < \varepsilon_1$, $1 + \varepsilon \phi^{(1)} \geq 1/2$. Integrating the inequality (3.64) over $(0, t)$ yields

$$\begin{aligned}
\|(u_R^\varepsilon, \phi_R^\varepsilon)(t)\|_\varepsilon^2 & \leq C \|(u_R^\varepsilon, \phi_R^\varepsilon)(0)\|_\varepsilon^2 + \int_0^t C_1 (1 + \varepsilon^2 \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2) (1 + \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2) ds \\
& \leq C \|(u_R^\varepsilon, \phi_R^\varepsilon)(0)\|_\varepsilon^2 + \int_0^t C_1 (1 + \varepsilon \tilde{C}) (1 + \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2) ds,
\end{aligned}$$

where C is an absolute constant.

Recall that C_1 depends on $\|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2$ through $\varepsilon \|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2$ and is nondecreasing. Let $C'_1 = C_1(1)$ and $C_2 > C \sup_{\varepsilon < 1} \|(u_R^\varepsilon, \phi_R^\varepsilon)(0)\|_\varepsilon^2$. For any arbitrarily given $\tau > 0$, we choose \tilde{C} sufficiently large such that $\tilde{C} > e^{4C'_1 \tau} (1 + C_2)(1 + C'_1)$. Then there exists $\varepsilon_0 > 0$ such that $\varepsilon \tilde{C} \leq 1$ for all $\varepsilon < \varepsilon_0$, we have

$$\sup_{0 \leq t \leq \tau} \|(u_R^\varepsilon, \phi_R^\varepsilon)(t)\|_\varepsilon^2 \leq e^{4C'_1 \tau} (C_2 + 1) < \tilde{C}. \tag{3.65}$$

In particular, we have the uniform bound for $(u_R^\varepsilon, \phi_R^\varepsilon)$

$$\sup_{0 \leq t \leq \tau} \|(u_R^\varepsilon, \phi_R^\varepsilon)(t)\|_{H^2}^2 + \varepsilon \|\partial_x^3 (u_R^\varepsilon, \phi_R^\varepsilon)(t)\|^2 + \varepsilon^2 \|\partial_x^4 \phi_R^\varepsilon(t)\|^2 \leq \tilde{C}.$$

On the other hand, by Lemma 3.1 and (3.65), we have

$$\sup_{0 \leq t \leq \tau} \|n_R^\varepsilon(t)\|_{H^2}^2 \leq \tilde{C}.$$

It is now standard to obtain uniform estimates independent of ε by the continuity method. The proof of Theorem 1.3 is complete for the case of $T_i = 0$. \square

Proof of Theorem 1.3. Recall that the case of $T_i > 0$ is proved in Section 2. The proof of Theorem 1.3 is complete. \square

APPENDIX A.

This appendix consists of two main parts. In the first one, we give a simple proof of Theorem 1.2, and in the second one, we derive the remainder system.

Proof of Theorem 1.2. We only give *a priori* estimate. Consider the equation (1.15) for $k = 2$. Let $\tau > 0$ be arbitrarily fixed and \tilde{s}_1 be sufficiently large as in Theorem 1.1. Note $G^{(1)}$ only depends on $n^{(1)} \in H^{\tilde{s}_1}$, which is assumed to be bounded in

$L^\infty(-\tau, \tau; H^{\tilde{s}_1})$. Multiplying the equation (1.15) with $n^{(2)}$ and integrating over \mathbb{R} , by integration by parts, the dispersive term vanishes and we have

$$\frac{1}{2} \frac{d}{dt} \|n^{(2)}\|^2 \leq C \|\partial_x n^{(1)}\|_{L^\infty} \|n^{(2)}\|^2 + C \|G^{(1)}\| \|n^{(2)}\|^2 \leq C \|n^{(2)}\|^2.$$

Higher order estimate is similar, and we then have a unique global solution for $n^{(2)}$. For $k = 3$, recalling that $G^{(2)}$ only depends on $n^{(1)}$ and $n^{(2)}$, we have the similar estimate and hence the global existence and uniqueness is obtained. The general case can be proved by induction. \square

Derivation of the remainder system (1.17). In the following, we derive the remainder system (1.17) for $(n_R^\varepsilon, u_R^\varepsilon, \phi_R^\varepsilon)$. From Theorem 1.1 and 1.2, we have the following systems:

$$\begin{aligned} (n^{(1)}, u^{(1)}, \phi^{(1)}) &\text{ satisfies (1.8) and (1.10)} \\ (n^{(2)}, u^{(2)}, \phi^{(2)}) &\text{ satisfies (1.11) and (1.13)} \\ (n^{(k)}, u^{(k)}, \phi^{(k)}) &\text{ satisfies (1.14) and (1.15), } k = 3, 4. \end{aligned} \tag{A.1}$$

By Theorem 1.1 and 1.2, the solutions $(n^{(k)}, u^{(k)}, \phi^{(k)}) \in H^{\tilde{s}_k}$ ($k = 1, 2, 3, 4$) are global when \tilde{s}_k is sufficiently large. However, the systems such as (1.9) and (1.12) are convenient for us to derive the remainder system. Therefore, we first remark that from (A.1) we can derive (1.9) and higher order counterparts. Indeed, from (1.11a), we have exactly (1.9a). Differentiating (1.11b) with respect to x , we have exactly (1.9b). From (1.10), we subtract V times (1.9a) and $T_e/(4\pi e\bar{n}M)$ times (1.9c), we get (1.9b). Similarly, we can obtain (1.12) and the system (S_3) for the coefficients of ε^4 .

The coefficients of ε^4 . Setting the coefficient of ε^4 to be 0, we obtain

$$\begin{cases} \partial_t n^{(3)} - V \partial_x n^{(4)} + \partial_x u^{(4)} + \partial_x \left(\sum_{1 \leq i, j \leq 3; i+j=4} n^{(i)} u^{(j)} \right) = 0 & \text{(A.2a)} \\ \partial_t u^{(3)} - V \partial_x u^{(4)} + \sum_{1 \leq i, j \leq 4; i+j=4} u^{(i)} \partial_x u^{(j)} + \frac{T_i}{M} \partial_x n^{(4)} \\ \quad - \frac{T_i}{M} [\partial_x (n^{(1)} n^{(3)}) + (n^{(2)} - (n^{(1)})^2) \partial_x n^{(2)} \\ \quad + ((n^{(1)})^3 - 2n^{(1)} n^{(2)}) \partial_x n^{(1)}] = -\frac{e}{M} \partial_x \phi^{(4)} & \text{(A.2b)} \\ \partial_x^2 \phi^{(3)} = 4\pi e \bar{n} [\kappa \phi^{(4)} + \frac{\kappa^2}{2!} (2\phi^{(1)} \phi^{(3)} + (\phi^{(2)})^2) \\ \quad + \frac{\kappa^3}{3!} (3(\phi^{(1)})^2 \phi^{(2)}) + \frac{\kappa^4}{4!} (\phi^{(1)})^4 - n^{(4)}], & \text{(A.2c)} \end{cases} \tag{S_3}$$

where we set $\kappa = e/T_e$ for simplification. Inserting the expansion (1.16) into the system (1.3), and then subtracting $\{\varepsilon \times (1.5) + \varepsilon^2 \times (1.9) + \varepsilon^3 \times (1.12) + \varepsilon^4 \times (A.2)\}$, we get the remainder system (1.17) for $(n_R^\varepsilon, u_R^\varepsilon, \phi_R^\varepsilon)$. The details of deriving (1.17c) are given below, while the other two equations (1.17a) and (1.17b) are similar and

omitted. It is easy to see that the remainder terms on the LHS is $\varepsilon^5 \partial_x^2 \phi^{(4)} + \varepsilon^4 \partial_x^2 \phi_R^\varepsilon$. By Taylor expansion, we have

$$\begin{aligned} e^{\kappa(\varepsilon\hat{\phi} + \varepsilon^3\phi_R^\varepsilon)} &= 1 + \frac{1}{1!}\kappa(\varepsilon\hat{\phi} + \varepsilon^3\phi_R^\varepsilon) + \cdots + \frac{1}{4!}\kappa^4(\varepsilon\hat{\phi} + \varepsilon^3\phi_R^\varepsilon)^4 \\ &\quad + \frac{1}{4!}\int_0^1 e^{\theta\kappa(\varepsilon\hat{\phi} + \varepsilon^3\phi_R^\varepsilon)}(1-\theta)^4(\kappa(\varepsilon\hat{\phi} + \varepsilon^3\phi_R^\varepsilon))^5 d\theta, \end{aligned} \quad (\text{A.3})$$

where $\varepsilon\hat{\phi} = \varepsilon\phi^{(1)} + \cdots + \varepsilon^4\phi^{(4)}$. Now, the constant 1 cancels with the 1 in n of (1.16). From (1.8), the coefficient of the ε order is also exactly canceled. Then by (1.9), (1.12) and (A.2), all the coefficients before $\varepsilon^0, \varepsilon^1, \varepsilon^2, \varepsilon^3$ and ε^4 vanish except the terms involving ϕ_R^ε . Therefore, the remainder on the RHS of (1.17c) is give by

$$\begin{aligned} &4\pi e\bar{n}\{\kappa\varepsilon^3\phi_R^\varepsilon + \frac{\kappa^2}{2!}[\varepsilon^6(\phi_R^\varepsilon)^2 + 2\varepsilon^4\hat{\phi}\phi_R^\varepsilon] + \frac{\kappa^3}{3!}[\varepsilon^9(\phi_R^\varepsilon)^3 + 3\varepsilon^7\hat{\phi}(\phi_R^\varepsilon)^2 + 3\varepsilon^5(\hat{\phi})^2\phi_R^\varepsilon] \\ &\quad + \frac{\kappa^4}{4!}[\varepsilon^{12}(\phi_R^\varepsilon)^4 + 4\varepsilon^{10}\hat{\phi}(\phi_R^\varepsilon)^3 + 6\varepsilon^8(\hat{\phi})^2(\phi_R^\varepsilon)^2 + 4\varepsilon^6(\hat{\phi})^3\phi_R^\varepsilon] - \varepsilon^3 n_R^\varepsilon + \varepsilon^5 \hat{R}(\varepsilon\phi_R^\varepsilon) + \varepsilon^5 R_1\}, \end{aligned}$$

where \hat{R} is the remainder terms corresponding to the last integral term of (A.3), and $R_1 = R_1(\phi^{(1)}, \phi^{(2)}, \phi^{(3)}, \phi^{(4)})$ involving only $(\phi^{(1)}, \phi^{(2)}, \phi^{(3)}, \phi^{(4)})$ corresponds to the first five terms on the RHS of (A.3). This remainder term can be further rewritten as

$$\begin{aligned} &\varepsilon^3(4\pi e\bar{n})\{\kappa\phi_R^\varepsilon + \kappa^2(\varepsilon\phi^{(1)} + \varepsilon^2\phi^{(2)})\phi_R^\varepsilon + \frac{\kappa^3}{2}\varepsilon^2(\phi^{(1)})^2\phi_R^\varepsilon \\ &\quad - n_R^\varepsilon + \varepsilon^2\frac{\kappa^2}{2!}(\sqrt{\varepsilon}\phi_R^\varepsilon)^2 + \varepsilon^2\hat{R}'(\varepsilon\phi_R^\varepsilon)\}, \end{aligned}$$

for some \hat{R}' depending on $\varepsilon\phi_R^\varepsilon$. After divided by ε^3 , the remainder equation for the Poisson equation is written as

$$\varepsilon\partial_x^2\phi_R^\varepsilon = 4\pi e\bar{n}\{\kappa\phi_R^\varepsilon + \varepsilon\kappa^2\phi^{(1)}\phi_R^\varepsilon - n_R^\varepsilon\} + \varepsilon^2\mathcal{R}_3,$$

where

$$\mathcal{R}_3 = [\frac{\kappa^2}{2!}(\varepsilon\phi_R^\varepsilon) + \kappa^2(\phi^{(2)} + \frac{\kappa}{2}(\phi^{(1)})^2)]\phi_R^\varepsilon + \hat{R}'(\varepsilon\phi_R^\varepsilon). \quad (\text{A.4})$$

The other two equations (1.17a) and (1.17b) can be derived similarly and we omit the details.

Lemma A.1. *For $\alpha = 0$, there exists some constant $C = C(\|\phi^{(i)}\|_{H^{\bar{s}_i}}, \varepsilon\|\phi_R^\varepsilon\|_{H^1})$ or when $\alpha = 1, \dots$ integers, there exists some constant $C = C(\|\phi^{(i)}\|_{H^{\bar{s}_i}}, \varepsilon\|\phi_R^\varepsilon\|_{H^\alpha})$ such that*

$$\begin{aligned} \|\mathcal{R}_3\|_{H^\alpha} &\leq C(\|\phi^{(i)}\|_{H^{\bar{s}_i}}, \varepsilon\|\phi_R^\varepsilon\|_{H^1})\|\phi_R^\varepsilon\|_{H^\alpha}, \quad \alpha = 0, 1, \\ \|\mathcal{R}_3\|_{H^\alpha} &\leq C(\|\phi^{(i)}\|_{H^{\bar{s}_i}}, \varepsilon\|\phi_R^\varepsilon\|_{H^\alpha})\|\phi_R^\varepsilon\|_{H^\alpha}, \quad \forall \alpha \geq 2. \end{aligned} \quad (\text{A.5})$$

$$\begin{aligned} \|\partial_t\mathcal{R}_3\|_{H^\alpha} &\leq C(\|\phi^{(i)}\|_{H^{\bar{s}_i}}, \varepsilon\|\phi_R^\varepsilon\|_{H^1})\|\partial_t\phi_R^\varepsilon\|_{H^\alpha}, \quad \alpha = 0, 1, \\ \|\partial_t\mathcal{R}_3\|_{H^\alpha} &\leq C(\|\phi^{(i)}\|_{H^{\bar{s}_i}}, \varepsilon\|\phi_R^\varepsilon\|_{H^\alpha})\|\partial_t\phi_R^\varepsilon\|_{H^\alpha}, \quad \forall \alpha \geq 2. \end{aligned} \quad (\text{A.6})$$

Recalling the fact that H^1 is an algebra, the estimate for Lemma A.1 is straightforward. The details are hence omitted. Combining Lemma 3.3 and Lemma 3.2, we have

Corollary A.1. *For any $\alpha = 0, 1, 2$, there exists some constant $C = C(\|\phi^{(i)}\|_{H^{\bar{s}_i}}, \varepsilon\|\phi_R^\varepsilon\|_{H^2})$ such that*

$$\varepsilon\|\varepsilon\partial_t\partial_x^\alpha\mathcal{R}_3\|^2 \leq C(\|\phi^{(i)}\|_{H^{\bar{s}_i}}, \varepsilon\|\phi_R^\varepsilon\|_{H^2})\|(u_R^\varepsilon, \phi_R^\varepsilon)\|_\varepsilon^2.$$

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